Marginal and joint reliability importance based on survival signature

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Abstract

Marginal and joint reliability importance measures have been found to be useful in optimal system design. Various importance measures have been defined and studied for a variety of system models. The results in the literature are mostly based on the assumption that the components within the system are independent or identical. The present paper is concerned with computation of marginal and joint reliability importance for a coherent system that consists of multiple types of dependent components. In particular, by utilizing the concept of survival signature, expressions for marginal and joint reliability importance measures are presented. We also introduce reliability importance for a system of which only the survival signature is known, which therefore can be regarded to be a black box system.

Keywords: Black box systems, Joint reliability importance, Marginal reliability importance, Survival signature.

1. Introduction

Knowing the relative importance of a component in a system is useful in design, improvement, and control of engineering systems. For these kind of purposes, various importance measures have been defined and studied in the reliability literature. According to Birnbaum [2], importance measures are categorized into three classes which are structure importance measures, reliability importance measures, and lifetime importance measures. Structural importance measures the relative importance of components with respect to their positions in a system and it needs knowledge only about the system's structure function. Reliability importance measures depend on both the structure of the system and the reliabilities of components. Lifetime importance measures depend on both the structure of the system and the component lifetime distributions. An extensive review of reliability importance measures is presented in Kuo and Zhu [17]. Besides classical importance

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measures, new measures have also been defined and used in optimal system design. For example, Wu and Coolen [23] proposed a cost-based importance measure for repairable systems; Zhai et al. [24] studied generalized moment-independent importance measures based on Minkowski distance; Borgonovo et al. [3] defined time independent importance measure that depends on the change in mean time to failure; Kvassay et al. [18] present a method to analyse component importance for multi-state systems.

To make more effective decisions in designing a system, it might be suitable to measure not only the effect of a single component on the system's performance but also to measure the interaction of two or more components in the system for their contribution to system reliability. We restrict attention in this paper to joint importance of two components, the presented theory can quite straightforwardly be generalized to more than two components. The joint effect of two components on the system's performance can be measured by the joint reliability importance (JRI) (Armstrong [1]). Using JRI, we can determine that one component is more or less important, or has the same importance when the other is functioning. Hagstrom [14] defined two components as reliability substitutes (complements) if the JRI is non-positive (non-negative). In particular, if $JRI > 0$, then one component becomes more important when the other is functioning (synergy); if $JRI < 0$, then one component becomes less important when the other is functioning (diminishing returns); and if $JRI = 0$, then one component's importance is unchanged by the functioning of the other (Armstrong [1]).

The problem of computing and evaluating JRI of two components has attracted considerable attention in the reliability literature. The JRI of two components in k-out-of-n:G systems has been studied in Hong et al. [16], Gao et al. [12], and Boushaba and Eryilmaz [4]. Gertsbakh and Shpungin [13] presented a combinatorial approach to compute JRI of two components in a coherent system that consists of independent and identical components. Rani et al. [21] studied conditional marginal and conditional joint reliability importance in series-parallel systems. Computational results on JRI have been presented in Eryilmaz [7], Zhu et al. [25], Zhu et al. [26], Zhu and Boushaba [27] for more general coherent systems such as linear m-consecutive-k-out-of-n:F system, and consecutive-k-within-m-out-of-n:F system. Eryilmaz et al. [9] presented a general formula for computing the JRI of two components in a binary coherent system that consists of exchangeable dependent components. In the exchangeable case, the components are dependent but they have the same distribution. That is, all components within the system are of the same type in terms of their failure time distribution, and the joint distribution of failure times of any r $(\leq n)$ of them is the same as for any other group of r components.

In this paper, we consider a more general and realistic case when the system is composed of $K \geq 2$ types of dependent components. Under this general setup, the random failure times of components of the same type are exchangeable dependent and the random failure times of components of different types are dependent, these concepts are explained in Section 2. Allowing the components' failure times to be dependent is potentially useful in numerous real-life situations, indeed the often used assumption of independence of components' failure times in the reliability literature may well frequently be made for mathematical convenience. As an example, consider a power generation system that consists of different types of generating units (components) having different capacities. If these units are subject to a common environmental random stress, or other common failure mode, then their random failure times are dependent. Other examples where dependence between the components in a system is important include load-sharing scenarios and situations where the failure of one component influences the functioning of other components.

The concept of survival signature has been found to be very useful to study reliability properties of such systems with components of multiple types, and multiple components of at least one of the types (see, e.g. Coolen and Coolen-Maturi [5]). By utilizing the idea behind the survival signature, we obtain expressions for the marginal and joint reliability importance of components in a coherent system that consists of multiple types of dependent components. The results of the present paper generalize and extend the results in Gertsbakh and Shpungin [13], Eryilmaz [8] and Eryilmaz et al. [9].

This paper is organized as follows. In Section 2, we present definitions and notation that will be used throughout the paper. Sections 3 and 4 present results on marginal and joint reliability importance measures, respectively. These results are illustrated via examples in Section 5. Section 6 presents a variation for the case that a system's structure is not known, instead one only knows the number of components of each type and the survival signature. It should be emphasized that the results presented for dependent components in this case of course also imply the results for the more basic scenario where the components of different types in the black-box system can be assumed to have independent failure times. Some concluding remarks are given in Section 7.

2. Definitions and notation

Consider a coherent system with in total n components of $K \geq 2$ types.. Let n_i denote the number of components of type i, $i = 1, 2, ..., K$, where $n = \sum_{i=1}^{K} n_i$. It is assumed that the random failure times of components of the same type are exchangeable dependent and the random failure times of components of different types are dependent. If $C_i(t)$ denotes the number of components of type i working at time t, then the survival function of the system can be written as

$$P\{T_S > t\} = \sum_{l_1=0}^{n_1} \cdots \sum_{l_K=0}^{n_K} \Phi(l_1, ..., l_K) P\{C_1(t) = l_1, ..., C_K(t) = l_K\}, \tag{1}$$

where $\Phi(l_1, ..., l_K)$ represents the survival signature and is defined by [5, 6]

$$\Phi(l_1, ..., l_K) = \frac{r_{n_1,...,n_K}(l_1, ..., l_K)}{\binom{n_1}{l_1} \cdots \binom{n_K}{l_K}}, \tag{2}$$

In equation (2), $r_{n_1,...,n_K}(l_1,...,l_K)$ denotes the number of path sets of the system including exactly l_1 components of type 1, ..., exactly l_K components of type K.

Let $T_j^{(i)}$ denote the failure time of the jth component of type $i, i = 1, 2, ..., K$. The assumption that two components of the same type, say components j_1 and j_2 of type k, have exchangeable dependent failure times means that the $T_{j_1}^{(k)}$ and $T_{j_2}^{(k)}$ have the same marginal probability distributions but they are typically not independent, so the joint probability distribution for these random quantities is not necessarily equal to the product of their marginal probability distributions. This reflects that information about one of these random quantities can affect the marginal probability distribution of the other one, and the exchangeability assumption for the failure times typically reflects that one considers these components to be 'similar' with regard to their failure times while functioning in the system. Perhaps the easiest way to interpret exchangeability is that, if one would learn that one of these two components had failed, it would be either one of them with equal probability 0.5. The assumption that two components of different types, say components j_1 and j_2 of types k_1 and k_2, respectively, have dependent failure times means that the joint probability distribution for $T_{j_1}^{(k_1)}$ and $T_{j_2}^{(k_2)}$ is not necessarily equal to the product of their marginal probability distributions, and these marginal probability distributions are not assumed to be equal (as these random quantities are not assumed to be exchangeable). This just reflects that information about one of these random quantities can affect the marginal probability distribution of the other one.

From Theorem 1 of Eryilmaz [10], the joint distribution of $C_1(t), ..., C_K(t)$ can be written as

$$P\left\{C_1(t) = l_1, ..., C_K(t) = l_K\right\} = \binom{n_1}{l_1} ... \binom{n_K}{l_K} S_{n_1,...,n_K}(t; l_1, ..., l_K),$$

where

$$\begin{aligned} S_{n_1,...,n_K}(t; l_1, ..., l_K) &= \sum_{i_1=0}^{n_1-l_1} \cdots \sum_{i_K=0}^{n_K-l_K} (-1)^{i_1+...+i_K} \binom{n_1-l_1}{i_1} ... \binom{n_K-l_K}{i_K} \\ &\times P\left\{T_1^{(1)} > t, ..., T_{l_1+i_1}^{(1)} > t, ..., T_1^{(K)} > t, ..., T_{l_K+i_K}^{(K)} > t\right\}. \end{aligned} \quad (3)$$

Thus if the joint survival function of components is given, then the joint distribution of $C_1(t), ..., C_K(t)$ can be easily calculated. For an illustration, suppose that $K = 2$, and the joint survival function of $T_1^{(1)}, ..., T_{n_1}^{(1)}, T_1^{(2)}, ..., T_{n_2}^{(2)}$ is given by

$$P\left\{T_1^{(1)} > t_1^{(1)}, ..., T_{n_1}^{(1)} > t_{n_1}^{(1)}, T_1^{(2)} > t_1^{(2)}, ..., T_{n_2}^{(2)} > t_{n_2}^{(2)}\right\} = \left[1 + \theta_1 \sum_{i=1}^{n_1} t_i^{(1)} + \theta_2 \sum_{i=1}^{n_2} t_i^{(2)}\right]^{-\alpha}, \quad (4)$$

$\theta_1, \theta_2, \alpha > 0$. Then

$$P\left\{T_1^{(1)} > t, ..., T_{l_1+i_1}^{(1)} > t, T_1^{(2)} > t, ..., T_{l_2+i_2}^{(2)} > t\right\} = \left[1 + \theta_1(l_1+i_1)t + \theta_2(l_2+i_2)t\right]^{-\alpha},$$

and hence

$$
\begin{aligned}
& P\left\{C_1(t)=l_1, C_2(t)=l_2\right\} \\
= & \binom{n_1}{l_1}\binom{n_2}{l_2} \sum_{i_1=0}^{n_1-l_1} \sum_{i_2=0}^{n_2-l_2}(-1)^{i_1+i_2}\binom{n_1-l_1}{i_1}\binom{n_2-l_2}{i_2} \\
& \times\left[1+\theta_1\left(l_1+i_1\right) t+\theta_2\left(l_2+i_2\right) t\right]^{-\alpha},
\end{aligned}
$$

for $l_1=0,1,...,n_1$ and $l_2=0,1,...,n_2$.

For a component with failure time T_i, Birnbaum [2] defined the importance of the ith component at time t by

$$
I_i(t)=P\left\{T_S>t \mid T_i>t\right\}-P\left\{T_S>t \mid T_i \leq t\right\}, \tag{5}
$$

for $i=1,...,n$.

The time dependent joint reliability importance (JRI) of two components is defined as

$$
\begin{aligned}
JRI(i, j) = & P\left\{T_S>t \mid T_i>t, T_j>t\right\}-P\left\{T_S>t \mid T_i>t, T_j \leq t\right\} \\
& -P\left\{T_S>t \mid T_i \leq t, T_j>t\right\}+P\left\{T_S>t \mid T_i \leq t, T_j \leq t\right\},
\end{aligned} \tag{6}
$$

for $t>0$ (see, e.g. Armstrong [1]).

Consider a component i which is type a, $a=1,2,...,K$. Let $r_{n_1,...,n_K}^{+i}(l_1,...,l_K)$ denote the number of path sets of the system including component i of type a, l_1 components of type 1,...,l_a-1 components of type a,...,l_K components of type K. Similarly, let $r_{n_1,...,n_K}^{-i}(l_1,...,l_K)$ be the number of path sets of the system that do not include component i of type a, and include l_1 components of type 1,...,l_a components of type a,...,l_K components of type K. It is easy to see that

$$
r_{n_1,...,n_K}(l_1,...,l_K)=r_{n_1,...,n_K}^{+i}(l_1,...,l_K)+r_{n_1,...,n_K}^{-i}(l_1,...,l_K). \tag{7}
$$

Consider two components i and j. Assume that the component i is of type a, and the component j is of type b. Let $r_{n_1,...,n_K}^{+i,+j}(l_1,...,l_K)$ be the number of path sets of the system including component i of type a, component j of type b, l_1 components of type 1,...,l_a-1 components of type a,...,l_b-1 components of type b,...,l_K components of type K. Denote by $r_{n_1,...,n_K}^{+i,-j}(l_1,...,l_K)$ the number of path sets of the system including component i of type a, do not include component j of type b, and include l_1 components of type 1,...,l_a-1 components of type a,...,l_b components of type b,...,l_K components of type K. The quantity $r_{n_1,...,n_K}^{-i,+j}(l_1,...,l_K)$ represents the number of path sets of the system that do not include component i of type a, including component j of type b, l_1 components of type 1,...,l_a components of type a,...,l_b-1 components of type b,...,l_K components of type K. Finally, let $r_{n_1,...,n_K}^{-i,-j}(l_1,...,l_K)$ be the number of path sets of the system that do not include component i of type a, do not include component j of type b, including l_1 components of type 1,...,l_a components of type a,...,l_b components of type b,...,l_K components of type K. We have

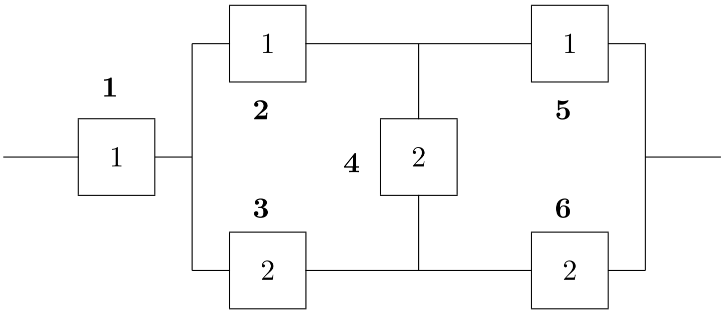


Figure 1: System with two types of components

$l_1\backslash l_2$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	2	1
3	1	3	3	1

Table 1: The numbers $r^{+i}_{n_1,n_2}(l_1,l_2)$ when $i=2$

the following relation.

$$\begin{aligned} r_{n_1,\ldots,n_K}(l_1,\ldots,l_K) &= r^{+i,+j}_{n_1,\ldots,n_K}(l_1,\ldots,l_K) + r^{+i,-j}_{n_1,\ldots,n_K}(l_1,\ldots,l_K) \\ &\quad + r^{-i,+j}_{n_1,\ldots,n_K}(l_1,\ldots,l_K) + r^{-i,-j}_{n_1,\ldots,n_K}(l_1,\ldots,l_K). \end{aligned} \tag{8}$$

It should also be noted that the following sum gives the total number of path sets of the system.

$$\sum_{l_1=0}^{n_1} \cdots \sum_{l_K=0}^{n_K} r_{n_1,\ldots,n_K}(l_1,\ldots,l_K).$$

For illustrating the numbers defined above, consider the system in Figure 1 which has been considered in Feng et al. [11]. The system has six components that belong to two types with $n_1 = 3$ and $n_2 = 3$. It can be easily checked that the system has 16 path sets. Tables 1 and 2 respectively include $r^{+2}_{n_1,n_2}(l_1,l_2)$ and $r^{-2}_{n_1,n_2}(l_1,l_2)$ for the component "2". The sum of the entries of Table 1 gives the total number of path sets including component "2". Tables 3 and 4 respectively include $r^{+1,+3}_{n_1,n_2}(l_1,l_2)$ and $r^{+1,-3}_{n_1,n_2}(l_1,l_2)$ for components "1" and "3". It should be noted that $r^{-1,+3}_{n_1,n_2}(l_1,l_2) = 0$ and $r^{-1,-3}_{n_1,n_2}(l_1,l_2) = 0$ for all l_1, l_2.

$l_1 \backslash l_2$	0	1	2	3
0	0	0	0	0
1	0	0	1	1
2	0	0	2	1
3	0	0	0	0

Table 2: The numbers $r^{-i}_{n_1,n_2}(l_1, l_2)$ when $i = 2$

$l_1 \backslash l_2$	0	1	2	3
0	0	0	0	0
1	0	0	1	1
2	0	0	3	2
3	0	1	2	1

Table 3: The numbers $r^{+i,+j}_{n_1,n_2}(l_1, l_2)$ when $i = 1$ and $j = 3$

3. Marginal reliability importance

In the following, we obtain a survival signature based expression for the marginal reliability importance (MRI) $I_i(t)$ when the system is formed by $K \geq 2$ types of components. Suppose that the ith component is of type a. Then, using the general property $P\{A \mid B\} = P\{A, B\} / P\{B\}$ for events A, B, and bringing the $C_k(t)$ into the argument, we get

$$P\left\{T_S > t \mid T_i^{(a)} > t\right\} = \frac{1}{P\left\{T_i^{(a)} > t\right\}} \sum \cdots \sum_{(l_1,\ldots,l_K)\in L_1} P\left\{T_S > t, T_i^{(a)} > t, C_1(t) = l_1, \ldots, C_K(t) = l_K\right\},$$

where $L_1 = \{(l_1, \ldots, l_K) : 0 < l_a \leq n_a; 0 \leq l_m \leq n_m, m \neq a\}$. Now we can rewrite the joint probability within the summation on the right-hand side by effectively using the general property

$l_1 \backslash l_2$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	1	0
3	1	2	1	0

Table 4: The numbers $r^{+i,-j}_{n_1,n_2}(l_1, l_2)$ when $i = 1$ and $j = 3$

$P\{A,B\} = P\{A \mid B\} P\{B\}$, leading to

$$\begin{aligned}
P\left\{T_S > t \mid T_i^{(a)} > t\right\} &= \frac{1}{P\left\{T_i^{(a)} > t\right\}} \sum \cdots \sum_{(l_1,...,l_K)\in L_1} P\left\{T_S > t \mid T_i^{(a)} > t,\right. \\
&\quad \left. C_1(t) = l_1, ..., C_K(t) = l_K\right\} P\left\{T_i^{(a)} > t, C_1(t) = l_1, ..., C_K(t) = l_K\right\} \\
&= \frac{1}{P\left\{T_i^{(a)} > t\right\}} \sum \cdots \sum_{(l_1,...,l_K)\in L_1} \Phi^{+i}(l_1,...,l_K) P\left\{T_i^{(a)} > t,\right. \\
&\quad \left. C_1(t) = l_1, ..., C_K(t) = l_K\right\},
\end{aligned}$$

where

$$\Phi^{+i}(l_1,...,l_K) = \frac{r^{+i}_{n_1,...,n_K}(l_1,...,l_K)}{\binom{n_a-1}{l_a-1}\prod_{m\neq a}\binom{n_m}{l_m}},$$

and $r^{+i}_{n_1,...,n_K}(l_1,...,l_K)$ denotes the number of path sets of the system including component i of type a, l_1 components of type 1,...,$l_a - 1$ components of type a,...,l_K components of type K. Because

$$\begin{aligned}
&P\left\{T_i^{(a)} > t, C_1(t) = l_1, ..., C_K(t) = l_K\right\} \\
&= \binom{n_a - 1}{l_a - 1} \prod_{m\neq a} \binom{n_m}{l_m} S_{n_1,...,n_K}(t; l_1, ..., l_K),
\end{aligned}$$

we obtain

$$\begin{aligned}
P\left\{T_S > t \mid T_i^{(a)} > t\right\} &= \frac{1}{P\left\{T_i^{(a)} > t\right\}} \sum \cdots \sum_{(l_1,...,l_K)\in L_1} \Phi^{+i}(l_1,...,l_K) \\
&\quad \times \binom{n_a - 1}{l_a - 1} \prod_{m\neq a} \binom{n_m}{l_m} S_{n_1,...,n_K}(t; l_1, ..., l_K) \qquad (9)
\end{aligned}$$

Similarly, it can be shown that

$$\begin{aligned}
P\left\{T_S > t \mid T_i^{(a)} \leq t\right\} &= \frac{1}{P\left\{T_i^{(a)} \leq t\right\}} \sum \cdots \sum_{(l_1,...,l_K)\in L_2} \Phi^{-i}(l_1,...,l_K) \\
&\quad \times \binom{n_a - 1}{l_a} \prod_{m\neq a} \binom{n_m}{l_m} S_{n_1,...,n_K}(t; l_1, ..., l_K), \qquad (10)
\end{aligned}$$

where $L_2 = \{(l_1,...,l_K) : 0 \leq l_a < n_a; 0 \leq l_m \leq n_m, m \neq a\}$,

$$\Phi^{-i}(l_1,...,l_K) = \frac{r^{-i}_{n_1,...,n_K}(l_1,...,l_K)}{\binom{n_a-1}{l_a}\prod_{m\neq a}\binom{n_m}{l_m}},$$

and $r^{-i}_{n_1,...,n_K}(l_1,...,l_K)$ is the number of path sets of the system that do not include component i of type a, l_1 components of type 1,...,l_a components of type a,...,l_K components of type K. From equations (9) and (10), the MRI of component i of type a is obtained as

$$\begin{aligned} I_i(t) &= \frac{1}{P\left\{T_i^{(a)} > t\right\}} \sum_{(l_1,...,l_K)\in L_1} \cdots \sum \Phi^{+i}(l_1,...,l_K) \\ &\quad \times \binom{n_a-1}{l_a-1} \prod_{m\neq a} \binom{n_m}{l_m} S_{n_1,...,n_K}(t;l_1,...,l_K) \\ &\quad - \frac{1}{P\left\{T_i^{(a)} \leq t\right\}} \sum_{(l_1,...,l_K)\in L_2} \cdots \sum \Phi^{-i}(l_1,...,l_K) \\ &\quad \times \binom{n_a-1}{l_a} \prod_{m\neq a} \binom{n_m}{l_m} S_{n_1,...,n_K}(t;l_1,...,l_K). \end{aligned} \tag{11}$$

4. Joint reliability importance

In the following, we present an expression for the JRI of two components when they belong to two different groups, and when they are of same type.

4.1. JRI between two components belonging different groups

Assume first that the component i is of type a, and the component j is of type b. Then the JRI between components i and j is defined by

$$\begin{aligned} JRI(i,j) &= P\left\{T_S > t \mid T_i^{(a)} > t, T_j^{(b)} > t\right\} - P\left\{T_S > t \mid T_i^{(a)} > t, T_j^{(b)} \leq t\right\} \\ &\quad - P\left\{T_S > t \mid T_i^{(a)} \leq t, T_j^{(b)} > t\right\} + P\left\{T_S > t \mid T_i^{(a)} \leq t, T_j^{(b)} \leq t\right\}. \end{aligned} \tag{12}$$

Let us first consider the conditional probability $P\left\{T_S > t \mid T_i^{(a)} > t, T_j^{(b)} > t\right\}$.

$$\begin{aligned} & P\left\{T_S > t \mid T_i^{(a)} > t, T_j^{(b)} > t\right\} \\ = & \frac{1}{P\left\{T_i^{(a)} > t, T_j^{(b)} > t\right\}} \sum_{(l_1,...,l_K)\in U_1} \cdots \sum P\left\{T_S > t \mid T_i^{(a)} > t, T_j^{(b)} > t,\right. \\ & \left. C_1(t) = l_1, ..., C_K(t) = l_K\right\} P\left\{T_i^{(a)} > t, T_j^{(b)} > t, C_1(t) = l_1, ..., C_K(t) = l_K\right\} \\ = & \frac{1}{P\left\{T_i^{(a)} > t, T_j^{(b)} > t\right\}} \sum_{(l_1,...,l_K)\in U_1} \cdots \sum \Phi^{+i,+j}(l_1,...,l_K) P\left\{T_i^{(a)} > t, T_j^{(b)} > t,\right. \\ & \left. C_1(t) = l_1, ..., C_K(t) = l_K\right\}, \end{aligned}$$

where $U_1 = \{(l_1, ..., l_K) : 0 < l_a \leq n_a, 0 < l_b \leq n_b, 0 \leq l_m \leq n_m, m \neq a, b\}$,

$$\Phi^{+i,+j}(l_1, ..., l_K) = \frac{r^{+i,+j}_{n_1,...,n_K}(l_1, ..., l_K)}{\binom{n_a-1}{l_a-1}\binom{n_b-1}{l_b-1}\prod_{m\neq a,b}\binom{n_m}{l_m}},$$

and $r^{+i,+j}_{n_1,...,n_K}(l_1, ..., l_K)$ is the number of path sets of the system including component i of type a, component j of type b, l_1 components of type 1,...,$l_a - 1$ components of type a,...,$l_b - 1$ components of type b,...,l_K components of type K. Because

$$\begin{aligned} & P\left\{T_i^{(a)} > t, T_j^{(b)} > t, C_1(t) = l_1, ..., C_K(t) = l_K\right\} \\ = & \binom{n_a - 1}{l_a - 1}\binom{n_b - 1}{l_b - 1}\prod_{m\neq a,b}\binom{n_m}{l_m}S_{n_1,...,n_K}(t; l_1, ..., l_K), \end{aligned}$$

we obtain

$$\begin{aligned} & P\left\{T_S > t \mid T_i^{(a)} > t, T_j^{(b)} > t\right\} \\ = & \frac{1}{P\left\{T_i^{(a)} > t, T_j^{(b)} > t\right\}}\sum\cdots\sum_{(l_1,...,l_K)\in U_1}\Phi^{+i,+j}(l_1, ..., l_K) \\ & \times\binom{n_a - 1}{l_a - 1}\binom{n_b - 1}{l_b - 1}\prod_{m\neq a,b}\binom{n_m}{l_m}S_{n_1,...,n_K}(t; l_1, ..., l_K) \end{aligned} \tag{13}$$

Similarly, we can also obtain

$$\begin{aligned} & P\left\{T_S > t \mid T_i^{(a)} > t, T_j^{(b)} \leq t\right\} \\ = & \frac{1}{P\left\{T_i^{(a)} > t, T_j^{(b)} \leq t\right\}}\sum\cdots\sum_{(l_1,...,l_K)\in U_2}\Phi^{+i,-j}(l_1, ..., l_K) \\ & \times\binom{n_a - 1}{l_a - 1}\binom{n_b - 1}{l_b}\prod_{m\neq a,b}\binom{n_m}{l_m}S_{n_1,...,n_K}(t; l_1, ..., l_K), \end{aligned} \tag{14}$$

where $U_2 = \{(l_1, ..., l_K) : 0 < l_a \leq n_a, 0 \leq l_b < n_b, 0 \leq l_m \leq n_m, m \neq a, b\}$,

$$\Phi^{+i,-j}(l_1, ..., l_K) = \frac{r^{+i,-j}_{n_1,...,n_K}(l_1, ..., l_K)}{\binom{n_a-1}{l_a-1}\binom{n_b-1}{l_b}\prod_{m\neq a,b}\binom{n_m}{l_m}},$$

and $r^{+i,-j}_{n_1,...,n_K}(l_1, ..., l_K)$ is the number of path sets of the system including component i of type a, do not include component j of type b, l_1 components of type 1,...,$l_a - 1$ components of type a,...,l_b

components of type b,...,l_K components of type K.

$$\begin{aligned}
& P\left\{T_S > t \mid T_i^{(a)} \leq t, T_j^{(b)} > t\right\} \\
= \quad & \frac{1}{P\left\{T_i^{(a)} \leq t, T_j^{(b)} > t\right\}} \sum \cdots \sum_{(l_1,\ldots,l_K)\in U_3} \Phi^{-i,+j}(l_1, ..., l_K) \\
& \times \binom{n_a - 1}{l_a} \binom{n_b - 1}{l_b - 1} \prod_{m \neq a,b} \binom{n_m}{l_m} S_{n_1,\ldots,n_K}(t; l_1, ..., l_K) \qquad (15)
\end{aligned}$$

where $U_3 = \{(l_1, ..., l_K) : 0 \leq l_a < n_a, 0 < l_b \leq n_b, 0 \leq l_m \leq n_m, m \neq a, b\}$,

$$\Phi^{-i,+j}(l_1, ..., l_K) = \frac{r^{-i,+j}_{n_1,\ldots,n_K}(l_1, ..., l_K)}{\binom{n_a-1}{l_a}\binom{n_b-1}{l_b-1}\prod_{m\neq a,b}\binom{n_m}{l_m}},$$

and $r^{-i,+j}_{n_1,\ldots,n_K}(l_1, ..., l_K)$ is the number of path sets of the system that do not include component i of type a, including component j of type b, l_1 components of type 1,...,l_a components of type a,...,$l_b - 1$ components of type b,...,l_K components of type K.

$$\begin{aligned}
& P\left\{T_S > t \mid T_i^{(a)} \leq t, T_j^{(b)} \leq t\right\} \\
= \quad & \frac{1}{P\left\{T_i^{(a)} \leq t, T_j^{(b)} \leq t\right\}} \sum \cdots \sum_{(l_1,\ldots,l_K)\in U_4} \Phi^{-i,-j}(l_1, ..., l_K) \\
& \times \binom{n_a - 1}{l_a} \binom{n_b - 1}{l_b} \prod_{m \neq a,b} \binom{n_m}{l_m} S_{n_1,\ldots,n_K}(t; l_1, ..., l_K) \qquad (16)
\end{aligned}$$

where $U_4 = \{(l_1, ..., l_K) : 0 \leq l_a < n_a, 0 \leq l_b < n_b, 0 \leq l_m \leq n_m, m \neq a, b\}$,

$$\Phi^{-i,-j}(l_1, ..., l_K) = \frac{r^{-i,-j}_{n_1,\ldots,n_K}(l_1, ..., l_K)}{\binom{n_a-1}{l_a}\binom{n_b-1}{l_b}\prod_{m\neq a,b}\binom{n_m}{l_m}},$$

and $r^{-i,-j}_{n_1,\ldots,n_K}(l_1, ..., l_K)$ is the number of path sets of the system that do not include component i of type a, do not include component j of type b, including l_1 components of type 1,...,l_a components of type a,...,l_b components of type b,...,l_K components of type K.

Thus the JRI between components i (type a) and j (type b) can be computed using equations (13)-(16) in equation (12).

4.2. JRI between two components belonging the same group

Now, assume that the components i and j are of same type, say a, $1 \le a \le K$. Then

$$\begin{aligned} & P\left\{T_S > t \mid T_i^{(a)} > t, T_j^{(a)} > t\right\} \\ = \quad & \frac{1}{P\left\{T_i^{(a)} > t, T_j^{(a)} > t\right\}} \sum \cdots \sum_{(l_1,\dots,l_K)\in U_1'} \Phi^{+i,+j}(l_1,...,l_K) \\ & \times \binom{n_a-2}{l_a-2} \prod_{m\neq a} \binom{n_m}{l_m} S_{n_1,\dots,n_K}(t; l_1,...,l_K), \end{aligned} \tag{17}$$

where $U_1' = \{(l_1,...,l_K) : 1 < l_a \le n_a, 0 \le l_m \le n_m, m \neq a\}$,

$$\Phi^{+i,+j}(l_1,...,l_K) = \frac{r^{+i,+j}_{n_1,\dots,n_K}(l_1,...,l_K)}{\binom{n_a-2}{l_a-2}\prod_{m\neq a}\binom{n_m}{l_m}},$$

and $r^{+i,+j}_{n_1,\dots,n_K}(l_1,...,l_K)$ is the number of path sets of the system including component i of type a, component j of type a, l_1 components of type 1,...,$l_a - 2$ components of type a,...,l_K components of type K.

$$\begin{aligned} & P\left\{T_S > t \mid T_i^{(a)} > t, T_j^{(a)} \le t\right\} \\ = \quad & \frac{1}{P\left\{T_i^{(a)} > t, T_j^{(a)} \le t\right\}} \sum \cdots \sum_{(l_1,\dots,l_K)\in U_2'} \Phi^{+i,-j}(l_1,...,l_K) \\ & \times \binom{n_a-2}{l_a-1} \prod_{m\neq a} \binom{n_m}{l_m} S_{n_1,\dots,n_K}(t; l_1,...,l_K), \end{aligned} \tag{18}$$

where $U_2' = \{(l_1,...,l_K) : 0 < l_a < n_a, 0 \le l_m \le n_m, m \neq a\}$,

$$\Phi^{+i,-j}(l_1,...,l_K) = \frac{r^{+i,-j}_{n_1,\dots,n_K}(l_1,...,l_K)}{\binom{n_a-2}{l_a-1}\prod_{m\neq a}\binom{n_m}{l_m}},$$

and $r^{+i,-j}_{n_1,\dots,n_K}(l_1,...,l_K)$ is the number of path sets of the system including component i of type a, do not include component j of type a, l_1 components of type 1,...,$l_a - 1$ components of type a,...,l_K

components of type K.

$$\begin{aligned}
& P\left\{T_S > t \mid T_i^{(a)} \le t, T_j^{(a)} > t\right\} \\
= \quad & \frac{1}{P\left\{T_i^{(a)} \le t, T_j^{(a)} > t\right\}} \sum \cdots \sum_{(l_1,...,l_K)\in U_3'} \Phi^{-i,+j}(l_1,...,l_K) \\
& \times \binom{n_a-2}{l_a-1} \prod_{m\neq a} \binom{n_m}{l_m} S_{n_1,...,n_K}(t; l_1,...,l_K) \qquad (19)
\end{aligned}$$

where $U_3' = \{(l_1,...,l_K) : 0 < l_a < n_a, 0 \le l_m \le n_m, m \ne a\}$,

$$\Phi^{-i,+j}(l_1,...,l_K) = \frac{r^{-i,+j}_{n_1,...,n_K}(l_1,...,l_K)}{\binom{n_a-2}{l_a-1}\prod_{m\neq a}\binom{n_m}{l_m}},$$

and $r^{-i,+j}_{n_1,...,n_K}(l_1,...,l_K)$ is the number of path sets of the system that do not include component i of type a, including component j of type a, l_1 components of type 1,...,$l_a - 1$ components of type a,...,l_K components of type K.

$$\begin{aligned}
& P\left\{T_S > t \mid T_i^{(a)} \le t, T_j^{(a)} \le t\right\} \\
= \quad & \frac{1}{P\left\{T_i^{(a)} \le t, T_j^{(a)} \le t\right\}} \sum \cdots \sum_{(l_1,...,l_K)\in U_4'} \Phi^{-i,-j}(l_1,...,l_K) \\
& \times \binom{n_a-2}{l_a} \prod_{m\neq a} \binom{n_m}{l_m} S_{n_1,...,n_K}(t; l_1,...,l_K) \qquad (20)
\end{aligned}$$

where $U_4' = \{(l_1,...,l_K) : 0 \le l_a < n_a - 1, 0 \le l_m \le n_m, m \ne a\}$,

$$\Phi^{-i,-j}(l_1,...,l_K) = \frac{r^{-i,-j}_{n_1,...,n_K}(l_1,...,l_K)}{\binom{n_a-2}{l_a}\prod_{m\neq a}\binom{n_m}{l_m}},$$

and $r^{-i,-j}_{n_1,...,n_K}(l_1,...,l_K)$ is the number of path sets of the system that do not include component i of type a, do not include component j of type a, including l_1 components of type 1,...,l_a components of type a,...,l_K components of type K.

Thus the JRI between components i (type a) and j (type a) can be computed using equations (17)-(20).

5. Illustrative examples

In this section, we compute MRI and JRI of components in the system given in Figure 1 when the joint survival function of components' lifetimes follow the model (4). To compute MRI of component i, we need the coefficients $\Phi^{+i}(l_1, l_2)$ and $\Phi^{-i}(l_1, l_2)$. For an illustration, we compute

$l_1 \backslash l_2$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	$\frac{1}{3}$	$\frac{1}{2}$
3	1	1	1	1

Table 5: The coefficients $\Phi^{+i}(l_1, l_2)$ when $i = 2$

$l_1 \backslash l_2$	0	1	2	3
0	0	0	0	0
1	0	0	$\frac{1}{6}$	$\frac{1}{2}$
2	0	0	$\frac{2}{3}$	1
3	0	0	0	0

Table 6: The coefficients $\Phi^{-i}(l_1, l_2)$ when $i = 2$

these coefficients for the component "2" in Tables 5 and 6. Manifestly,

$$\Phi^{+2}(l_1, l_2) = \frac{r^{+2}_{n_1,n_2}(l_1, l_2)}{\binom{2}{l_1-1}\binom{3}{l_2}},$$

$l_1 = 1, 2, 3$; $l_2 = 0, 1, 2, 3$.and

$$\Phi^{-2}(l_1, l_2) = \frac{r^{-2}_{n_1,n_2}(l_1, l_2)}{\binom{2}{l_1}\binom{3}{l_2}},$$

for $l_1 = 0, 1, 2$; $l_2 = 0, 1, 2, 3$.

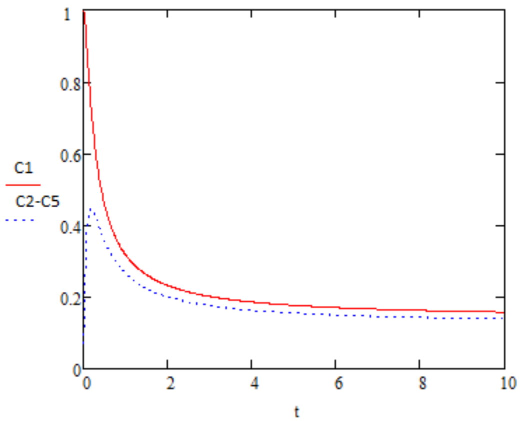
Because the component "2" is of type 1, its MRI can then be computed from

$$\begin{aligned} I_2(t) &= \frac{1}{P\left\{T_2^{(1)} > t\right\}} \sum_{l_1=1}^{3} \sum_{l_2=0}^{3} \Phi^{+2}(l_1, l_2) \binom{2}{l_1 - 1} \binom{3}{l_2} S_{n_1,n_2}(t; l_1, l_2) \\ &\quad - \frac{1}{P\left\{T_2^{(1)} \leq t\right\}} \sum_{l_1=0}^{2} \sum_{l_2=0}^{3} \Phi^{-2}(l_1, l_2) \binom{2}{l_1} \binom{3}{l_2} S_{n_1,n_2}(t; l_1, l_2), \end{aligned}$$

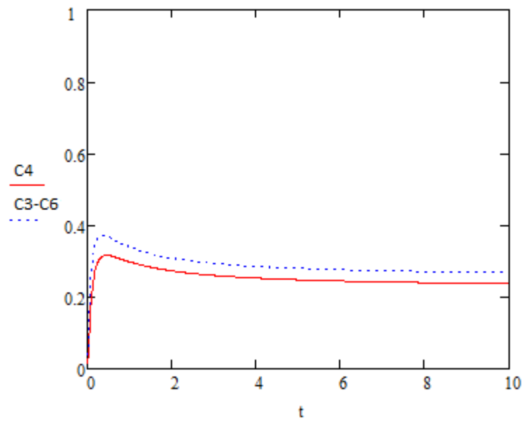
where $P\left\{T_2^{(1)} > t\right\} = (1 + \theta_1 t)^{-\alpha}$, and

$$S_{n_1,n_2}(t; l_1, l_2) = \sum_{i_1=0}^{n_1-l_1} \sum_{i_2=0}^{n_2-l_2} (-1)^{i_1+i_2} \binom{n_1 - l_1}{i_1} \binom{n_2 - l_2}{i_2} \left[1 + \theta_1(l_1 + i_1)t + \theta_2(l_2 + i_2)t\right]^{-\alpha}.$$

In Figure 2, we plot MRI of all components as a function of t when $\theta_1 = 1, \theta_2 = 2$ and $\alpha = 2$. Clearly, the components "2" and "5", and "3" and "6" have the same MRI values. Figure 3 plots MRI of all components when $\theta_1 = 2, \theta_2 = 1$ and $\alpha = 2$. Although the MRI of the component "1" is nonincreasing in t, the MRI of other components first increase until a specific time and then



(a) $\theta_1 = 1, \theta_2 = 2$ and $\alpha = 2$



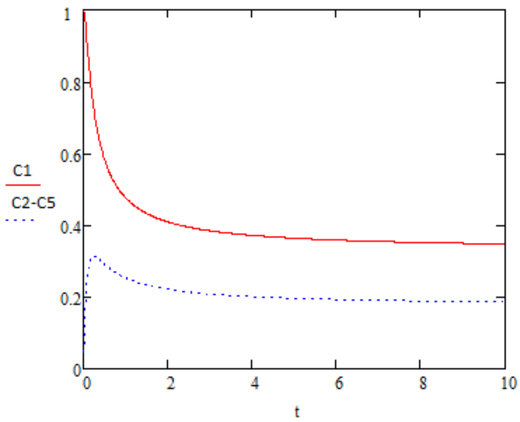
(b) $\theta_1 = 1, \theta_2 = 2$ and $\alpha = 2$

Figure 2: Marginal reliability importance

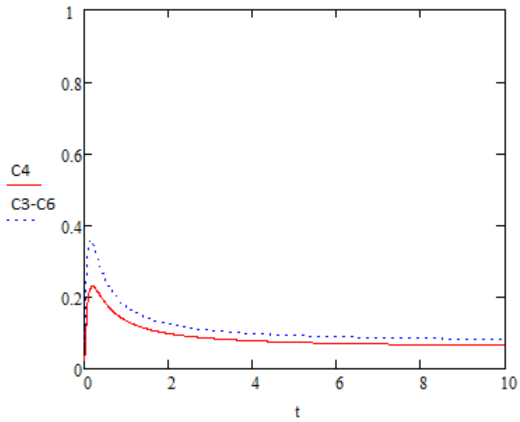
decrease. It should be noted that for the case when $\theta_1 = 1, \theta_2 = 2$ the lifetimes of components of type 1 are larger than the lifetime of components of type 2 in stochastic ordering, while the reverse is true when $\theta_1 = 2, \theta_2 = 1$. This leads e.g. to greater values for the MRI of component "1" as the MRI reflects the difference in system reliability between the situations where the specific component functions or has failed, yet this effect is strongly dependent on the functioning of the other components. Clearly, if the other components are likely to have failed at a time of interest, then functioning or not of the specific component of interest is unlikely to affect system functioning, hence low values of MRI for larger values of t tend to reflect that the other components are likely to have failed, leading to system failure independent of the status of the component of interest. For small values of t, MRI for a specific component may also be small, simply because all other components may still be very likely to function and failure of only the specific component may not lead to system failure. In this example, this latter effect is shown in the MRIs of all components except the critical component "1", which of course must function in order for the system to function.

Next, we compute and evaluate JRI values for the same system under the same joint survival model given by (4). It should be emphasized that the JRI is more difficult to interpret than the MRI, as it represents the interaction of the functioning or failing of two components on the system reliability. The easiest use of JRI is to consider its value for different pairs of components, where the pair of components with maximum JRI at a specific time can be interpreted as the two components whose joint functioning at the time considered is of most benefit compared to only either one of these two components functioning.

To compute JRI between components i and j, we need the coefficients $\Phi^{+i,+j}(l_1, l_2)$, $\Phi^{+i,-j}(l_1, l_2)$, $\Phi^{-i,+j}(l_1, l_2)$ and $\Phi^{-i,-j}(l_1, l_2)$. For an illustration, we compute these coefficients when $i = 1$ and $j = 3$. The results are presented in Tables 7 and 8. Clearly, $\Phi^{-1,+3}(l_1, l_2) = 0$ and $\Phi^{-1,-3}(l_1, l_2) = 0$ for all values of l_1 and l_2.



(a) $\theta_1 = 2, \theta_2 = 1$ and $\alpha = 2$



(b) $\theta_1 = 2, \theta_2 = 1$ and $\alpha = 2$

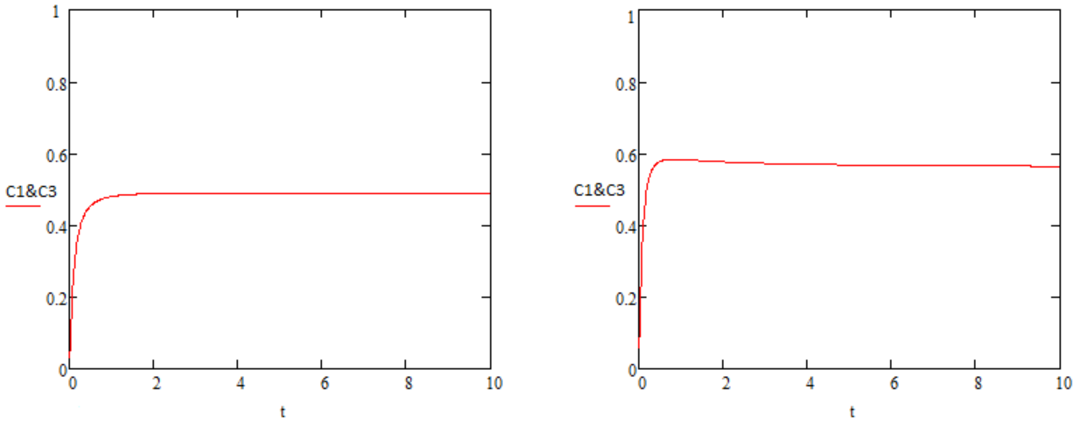
Figure 3: Marginal reliability importance

$l_1 \backslash l_2$	0	1	2	3
0	0	0	0	0
1	0	0	$\frac{1}{2}$	1
2	0	0	$\frac{3}{4}$	1
3	0	1	1	1

Table 7: The coefficients $\Phi^{+i,+j}(l_1, l_2)$ when $i = 1$ and $j = 3$

$l_1 \backslash l_2$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	$\frac{1}{2}$	0
3	1	1	1	0

Table 8: The coefficients $\Phi^{+i,-j}(l_1, l_2)$ when $i = 1$ and $j = 3$



(a) $\theta_1 = 1, \theta_2 = 2$ and $\alpha = 2$ (b) $\theta_1 = 2, \theta_2 = 1$ and $\alpha = 2$

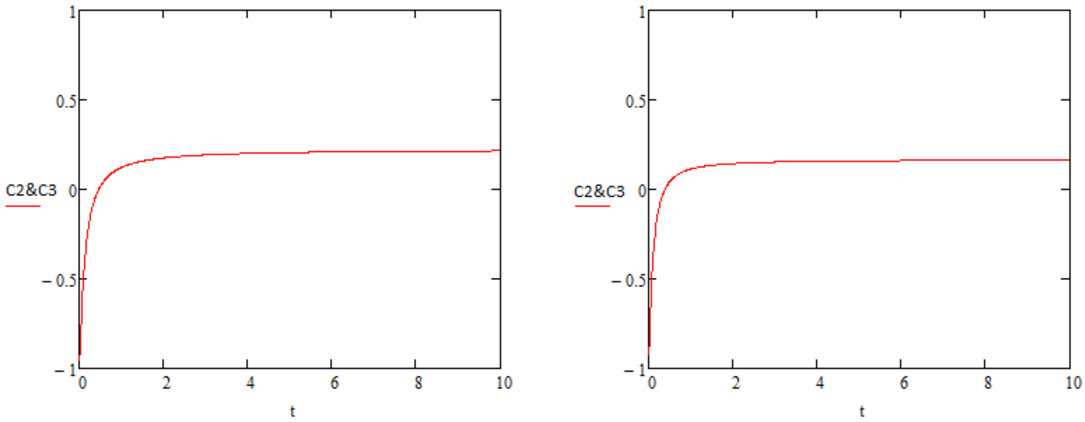
Figure 4: Joint reliability importance

Figure 4-7 plots JRI between pairs of components when $\theta_1 = 1, \theta_2 = 2$ and $\alpha = 2$, and when $\theta_1 = 2, \theta_2 = 1$ and $\alpha = 2$. From Figure 4, we observe that the components "1" and "3" are reliability complements. Components "1" and "3" are of different type, and the marginal lifetime distribution of component "1" ("3") is pareto with parameter θ_1 (θ_2). Thus when we shift the parameter values from $\theta_1 = 1, \theta_2 = 2$ to $\theta_1 = 2, \theta_2 = 1$ mean lifetime of component "1" decreases while the mean lifetime of component "3" increases. In this case, the components "1" and "3" become more reliability complements. As is clear from Figure 5, the components "2" and "3" are reliability substitutes until a specific time point, and they are reliability complements after that time. Components "1" and "4" become less reliability complements when parameter values are shifted from $\theta_1 = 1, \theta_2 = 2$ to $\theta_1 = 2, \theta_2 = 1$, i.e. when mean lifetime of component "1" decreases while the mean lifetime of component "4" increases.

Based on our calculations, we have observed that the two components which have largest MRI values are "1" and "2" (or "5") when t is fixed as $t_0 = 0.19$ and when $\theta_1 = 1, \theta_2 = 2$ and $\alpha = 2$ (see Figure 2a). At $t_0 = 0.19$ the JRI between "1" and "2" (or "5") is also the largest one among all the pairs (see Figure 6a), hence the importance of these components for the system reliability at that time point is not only maximal when considered per component, but also their joint functioning has the largest contribution to system reliability compared to scenarios where only one of these two components would function at that time.

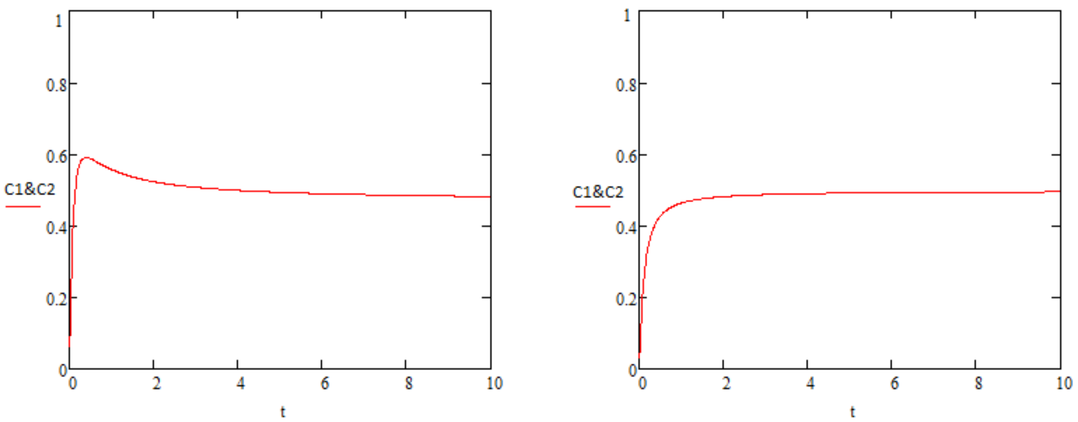
6. Reliability importance for black-box systems

The importance measures presented thus far in this paper use the survival signature to provide simple expressions and enable efficient calculations. However, by focusing on specific components, one needs not only the survival signature of the whole system but also of the system with the



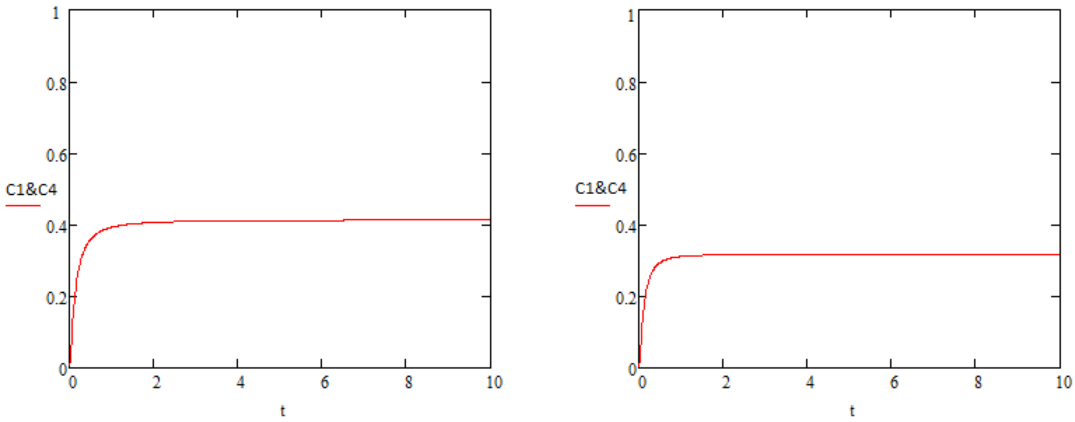
(a) $\theta_1 = 1, \theta_2 = 2$ and $\alpha = 2$ (b) $\theta_1 = 2, \theta_2 = 1$ and $\alpha = 2$

Figure 5: Joint reliability importance



(a) $\theta_1 = 1, \theta_2 = 2$ and $\alpha = 2$ (b) $\theta_1 = 2, \theta_2 = 1$ and $\alpha = 2$

Figure 6: Joint reliability importance



(a) $\theta_1 = 1, \theta_2 = 2$ and $\alpha = 2$ (b) $\theta_1 = 2, \theta_2 = 1$ and $\alpha = 2$

Figure 7: Joint reliability importance

condition that the specific component either functions or not. This implies that one must have access to more details about the system structure than provided by the single survival signature for the full system. For small systems, this is relatively easy to get, in particular when e.g. applying the efficient algorithm by Reed [22] for calculation of the survival signature. However, a main reason for the introduction of the survival signature has been the possibility to move quantification of system reliability to substantially larger systems and networks than when methods are used that explicitly require the use of the system structure function, which for larger real-world problems becomes impossible to specify. Hence, in this section we consider component importance measures based only on the survival signature of the full system. Such a view was recently also taken by Patelli et al. [20], who show how system failure times and corresponding estimates for the survival functions can be derived in a very efficient manner by simulation based on only the survival signature. This also fits with a view for larger systems or networks that, to the analyst and user of the system, the system design may be unknown, which may be occurring if use of a system is under a service-style contract where ownership remains with the system manufacturer. The manufacturer may not be willing to share detailed information about the system configuration, but may be willing to provide the survival signature. We first consider the marginal reliability importance for black box systems, followed by joint reliability importance for components of different types as well as for components of the same type. These are also related to the importance measures for specific components presented in the earlier sections. Throughout this section, we again consider the case where components of different types have dependent failure times, but of course the special case of independent failure times within such black-box systems is also covered by the results presented.

6.1. Marginal reliability importance for black-box systems

We are interested in the importance, for system reliability at time t, of a single component of type $a \in \{1, \ldots, K\}$. We assume, however, that the full system structure function is not known but only the survival signature is known. Hence, we consider the effect on the system survival function at time t of the information that one component of type a is known to function at this time, compared to the information that this component is known not to function at this time, where we do not know which specific component of type a this information is about and hence also not its specific role in the system. We assume that this specific component is selected by simple random sampling from the n_a components of this type in the system, independent of its functioning at time t. This assumption is fully in line with the assumed exchangeability of the failure times of all components of type a and the fact that no knowledge is assumed to be available about the system structure. One could e.g. interpret the information about the functioning of this component as resulting from interference with the system in the sense that one randomly selected component of type a has been replaced, for consideration of reliability at time t, by a similar component which is either certain to function or certain not to function.

We denote the survival function for the system, so the probability that the system functions at time t, given that one component of type a is known to function, by $P(T_S > t|a : 1)$, and the survival function for the system given that this one component is known to fail by $P(T_S > t|a : 0)$. The uncertainty about the number of components of each type that function at time t is again denoted by $C_m(t) \in \{0, 1, \ldots, n_m\}$ for all types $m \neq a$, while now there is only uncertainty about the remaining $n_a - 1$ components of type a for which we do not know if they function or not. We denote the number of these remaining components of type a which function at time t by $C'_a(t) \in \{0, 1, \ldots, n_a - 1\}$. This notation leads straightforwardly to

$$\begin{aligned} P\{T_S > t \mid a : 1\} &= \frac{1}{P\left\{T_i^{(a)} > t\right\}} \sum \cdots \sum_{(l_1,\ldots,l_K) \in L_1} \Phi(l_1, \ldots, l_K) \times \\ &\quad P\left\{ \bigcap_{m=1, m\neq a}^{K} \{C_m(t) = l_m\} \cap \left\{C'_a(t) = l_a - 1\right\} \cap T_i^{(a)} > t \right\}. \end{aligned}$$

where $L_1 = \{(l_1, ..., l_K) : 0 < l_a \leq n_a; 0 \leq l_m \leq n_m, m \neq a\}$. Because

$$\begin{aligned} & P\left\{ \bigcap_{m=1, m\neq a}^{K} \{C_m(t) = l_m\} \cap \left\{C'_a(t) = l_a - 1\right\} \cap T_i^{(a)} > t \right\} \\ &= \binom{n_a - 1}{l_a - 1} \prod_{m=1, m\neq a}^{K} \binom{n_m}{l_m} S_{n_1,\ldots,n_K}(t; l_1, \ldots, l_K), \end{aligned}$$

we obtain

$$P\left\{T_S > t \mid a:1\right\} = \frac{1}{P\left\{T_i^{(a)} > t\right\}} \sum \cdots \sum_{(l_1,\ldots,l_K)\in L_1} \Phi(l_1,\ldots,l_K) \times \binom{n_a-1}{l_a-1} \prod_{m=1, m\neq a}^{K} \binom{n_m}{l_m} S_{n_1,\ldots,n_K}(t; l_1,\ldots,l_K).$$

Similarly,

$$P\left\{T_S > t \mid a:0\right\} = \frac{1}{P\left\{T_i^{(a)} \leq t\right\}} \sum \cdots \sum_{(l_1,\ldots,l_K)\in L_2} \Phi(l_1,\ldots,l_K) \times \binom{n_a-1}{l_a} \prod_{m=1, m\neq a}^{K} \binom{n_m}{l_m} S_{n_1,\ldots,n_K}(t; l_1,\ldots,l_K).$$

where $L_2 = \{(l_1, ..., l_K) : 0 \leq l_a < n_a; 0 \leq l_m \leq n_m, m \neq a\}$.

We denote the marginal reliability importance at time t for a component of type a for which its exact location in the system is not known and with information about the system structure only available in the form of the survival signature, by $RI_a(t)$. This can be defined as

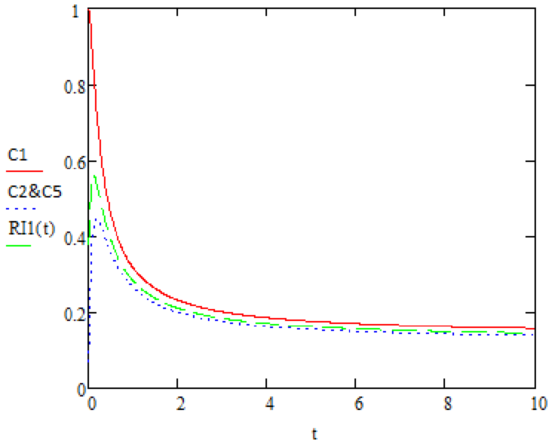
$$RI_a(t) = P\left\{T_S > t \mid a:1\right\} - P\left\{T_S > t \mid a:0\right\}.$$

The marginal reliability importance for an unspecified component of type a is logically related to the marginal reliability importances of all components of type a, due to the fact that the unspecified component is assumed to be any one of the n_a components of this type with equal probability. This leads straightforwardly to $RI_a(t)$ being equal to the average of the $I_i(t)$, as presented in Section 3, for all components i of type a. Figure 8a (8b) plots marginal importances of all components of type 1 (type 2) along with $I_1(t)$ ($I_2(t)$) for the system in Figure 1 when $\theta_1 = 1, \theta_2 = 2$ and $\alpha = 2$.

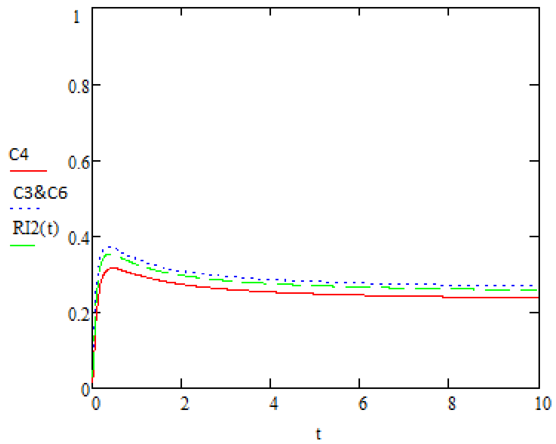
6.2. Joint (different types) reliability importance for black-box systems

We now consider the joint reliability importance for two unspecified components of different types, say type a and b. These components are again assumed to be drawn by simple random sampling from all components of type a and of type b, respectively. Additional notation used below straightforwardly generalizes notation from the previous section. Anologously to Equation (12), the joint reliability importance can be derived by

$$\begin{aligned} JRI(a,b) &= P\left(T_S > t \mid a:1, b:1\right) - P\left(T_S > t \mid a:1, b:0\right) - \\ &\quad P\left(T_S > t \mid a:0, b:1\right) + P\left(T_S > t \mid a:0, b:0\right) \end{aligned}$$



(a) Marginal importances of all components of type 1 and $RI_1(t)$



(b) Marginal importances of all components of type 2 and $RI_2(t)$

Figure 8: Black box marginal importance measures

The uncertainty about the number of components of each type that function at time t is again denoted by $C_m(t) \in \{0, 1, \dots, n_m\}$ for all types $m \notin \{a, b\}$, while now there is only uncertainty about the remaining $n_a - 1$ and $n_b - 1$ components of types a and b, respectively, for which we do not know if they function or not. We denote the number of these remaining components of types a and b which function at time t by $C'_a(t) \in \{0, 1, \dots, n_a - 1\}$ and $C'_b(t) \in \{0, 1, \dots, n_b - 1\}$, respectively. This leads to

$$P\{T_S > t \mid a:1, b:1\} = \frac{1}{P\left\{T_i^{(a)} > t, T_j^{(b)} > t\right\}} \sum \cdots \sum_{(l_1,\dots,l_K)\in U_1} \Phi(l_1,\dots,l_K) \times$$

$$P\left\{\bigcap_{m\notin\{a,b\}} \{C_m(t) = l_m\} \cap \left\{C'_a(t) = l_a - 1\right\} \cap \left\{C'_b(t) = l_b - 1\right\} \cap T_i^{(a)} > t \cap T_j^{(b)} > t\right\}.$$

where $U_1 = \{(l_1, ..., l_K) : 0 < l_a \leq n_a, 0 < l_b \leq n_b, 0 \leq l_m \leq n_m, m \neq a, b\}$,

Because

$$P\left\{\bigcap_{m=1, m\notin\{a,b\}}^{K} \{C_m(t) = l_m\} \cap \left\{C'_a(t) = l_a - 1\right\} \cap \left\{C'_b(t) = l_b - 1\right\} \cap T_i^{(a)} > t \cap T_j^{(b)} > t\right\}$$

$$= \binom{n_a - 1}{l_a - 1}\binom{n_b - 1}{l_b - 1} \prod_{m\notin\{a,b\}} \binom{n_m}{l_m} S_{n_1,\dots,n_K}(t; l_1, \dots, \dots, l_K),$$

we obtain

$$P\left\{T_S > t \mid a:1, b:1\right\} = \frac{1}{P\left\{T_i^{(a)} > t, T_j^{(b)} > t\right\}} \sum_{(l_1,\ldots,l_K)\in U_1} \cdots \sum \Phi(l_1,\ldots,l_K) \times \binom{n_a-1}{l_a-1}\binom{n_b-1}{l_b-1}\prod_{m\notin\{a,b\}}\binom{n_m}{l_m} S_{n_1,\ldots,n_K}(t;l_1,\ldots,l_K).$$

Similarly,

$$P\left\{T_S > t \mid a:1, b:0\right\} = \frac{1}{P\left\{T_i^{(a)} > t, T_j^{(b)} \leq t\right\}} \sum_{(l_1,\ldots,l_K)\in U_2} \cdots \sum \Phi(l_1,\ldots,l_K) \times \binom{n_a-1}{l_a-1}\binom{n_b-1}{l_b}\prod_{m\notin\{a,b\}}\binom{n_m}{l_m} S_{n_1,\ldots,n_K}(t;l_1,\ldots,l_K).$$

where $U_2 = \left\{(l_1,...,l_K) : 0 < l_a \leq n_a, 0 \leq l_b < n_b, 0 \leq l_m \leq n_m, m \neq a,b\right\}$.

$$P\left\{T_S > t \mid a:0, b:1\right\} = \frac{1}{P\left\{T_i^{(a)} \leq t, T_j^{(b)} > t\right\}} \sum_{(l_1,\ldots,l_K)\in U_3} \cdots \sum \Phi(l_1,\ldots,l_K) \times \binom{n_a-1}{l_a}\binom{n_b-1}{l_b-1}\prod_{m\notin\{a,b\}}\binom{n_m}{l_m} S_{n_1,\ldots,n_K}(t;l_1,\ldots,l_K).$$

where $U_3 = \left\{(l_1,...,l_K) : 0 \leq l_a < n_a, 0 < l_b \leq n_b, 0 \leq l_m \leq n_m, m \neq a,b\right\}$.

$$P\left\{T_S > t \mid a:0, b:0\right\} = \frac{1}{P\left\{T_i^{(a)} \leq t, T_j^{(b)} \leq t\right\}} \sum_{(l_1,\ldots,l_K)\in U_4} \cdots \sum \Phi(l_1,\ldots,l_K) \times \binom{n_a-1}{l_a}\binom{n_b-1}{l_b}\prod_{m\notin\{a,b\}}\binom{n_m}{l_m} S_{n_1,\ldots,n_K}(t;l_1,\ldots,l_K).$$

where $U_4 = \left\{(l_1,...,l_K) : 0 \leq l_a < n_a, 0 \leq l_b < n_b, 0 \leq l_m \leq n_m, m \neq a,b\right\}$.

It can be shown that $JRI(a,b)$ is again the average of all the joint reliability importance measures $JRI(i,j)$, as presented in Section 4, where i represents any component of type a and j any component of type b.

6.3. Joint (same type) reliability importance for black-box systems

We assume here that the two components considered, i and j, are of same type, say a, $1 \leq a \leq K$, where again nothing more is known about which specific components they are or their role in the system, so they can be regarded as being selected from all components of type a by simple random sampling (without replacement). The uncertainty about the number of components of each type that function at time t is again denoted by $C_m(t) \in \{0, 1, \ldots, n_m\}$ for all types $m \neq a$, while now there is only uncertainty about the remaining $n_a - 2$ components of type a for which we do not know if they function or not. We denote the number of these remaining components of type a which function at time t by $C''_a(t) \in \{0, 1, \ldots, n_a - 2\}$. The results below include some further new notation which is in line with earlier notation, where $a : 2$ denotes the event that both the considered components of type a function and $a : 1$ that one of these functions and the other does not, again of course without any knowledge of which specific component functions or not. The joint reliability importance can now be derived by

$$JRI(a, a) = P\left(T_S > t \mid a : 2\right) - 2P\left(T_S > t \mid a : 1\right) + P\left(T_S > t \mid a : 0\right)$$

We now have

$$\begin{aligned} P\left\{T_S > t \mid a : 2\right\} &= \frac{1}{P\left\{T_i^{(a)} > t, T_j^{(a)} > t\right\}} \sum_{(l_1,\ldots,l_K) \in U'_1} \cdots \sum \Phi(l_1, \ldots, l_K) \times \\ & P\left\{\bigcap_{m \neq a} \{C_m(t) = l_m\} \cap \left\{C''_a(t) = l_a - 2\right\} \cap T_i^{(a)} > t \cap T_j^{(a)} > t\right\}, \end{aligned}$$

where $U'_1 = \{(l_1, ..., l_K) : 1 < l_a \leq n_a, 0 \leq l_m \leq n_m, m \neq a\}$.

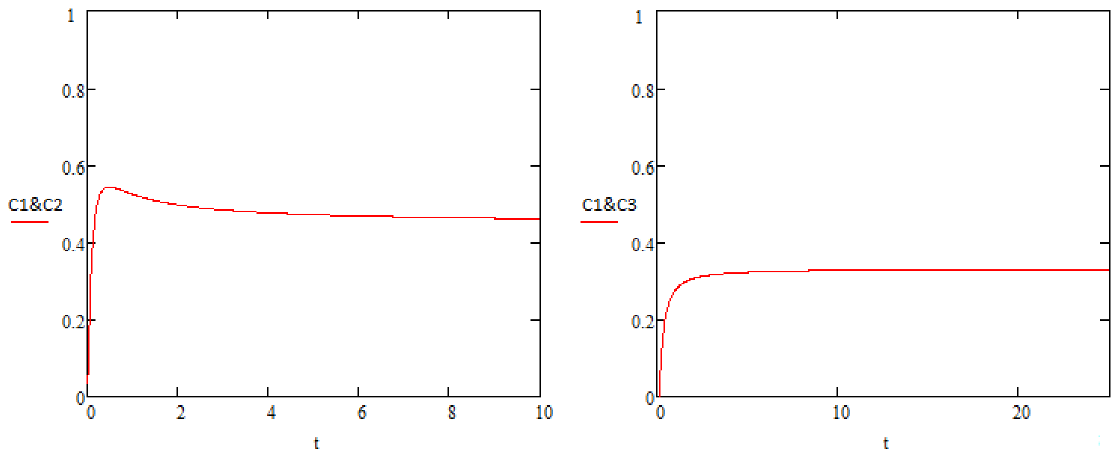
Because

$$\begin{aligned} & P\left\{\bigcap_{m \neq a} \{C_m(t) = l_m\} \cap \left\{C''_a(t) = l_a - 2\right\} \cap T_i^{(a)} > t \cap T_j^{(a)} > t\right\} \\ = & \binom{n_a - 2}{l_a - 2} \prod_{m \neq a} \binom{n_m}{l_m} S_{n_1,\ldots,n_K}(t; l_1, \ldots, l_K), \end{aligned}$$

we obtain

$$\begin{aligned} P\left\{T_S > t \mid a : 2\right\} &= \frac{1}{P\left\{T_i^{(a)} > t, T_j^{(a)} > t\right\}} \sum_{(l_1,\ldots,l_K) \in U'_1} \cdots \sum \Phi(l_1, \ldots, l_K) \times \\ & \binom{n_a - 2}{l_a - 2} \prod_{m \neq a} \binom{n_m}{l_m} S_{n_1,\ldots,n_K}(t; l_1, \ldots, l_K). \end{aligned}$$

Similarly,



(a) JRI(1,1) for the black box system when $\theta_1 = 1$, $\theta_2 = 2$ and $\alpha = 2$

(b) JRI(1,2) for the black box system when $\theta_1 = 1$, $\theta_2 = 2$ and $\alpha = 2$

Figure 9: Black box joint reliability importance

$$\begin{aligned} P\{T_S > t \mid a:1\} &= \frac{1}{P\left\{T_i^{(a)} > t, T_j^{(a)} \leq t\right\}} \sum \cdots \sum_{(l_1,\ldots,l_K) \in U_2'} \Phi(l_1,\ldots,l_K) \times \\ &\quad \binom{n_a - 2}{l_a - 1} \prod_{m \neq a} \binom{n_m}{l_m} S_{n_1,\ldots,n_K}(t; l_1,\ldots,l_K). \end{aligned}$$

where $U_2' = \{(l_1,...,l_K) : 0 < l_a < n_a, 0 \leq l_m \leq n_m, m \neq a\}$,

$$\begin{aligned} P\{T_S > t \mid a:0\} &= \frac{1}{P\left\{T_i^{(a)} \leq t, T_j^{(a)} \leq t\right\}} \sum \cdots \sum_{(l_1,\ldots,l_K) \in U_3'} \Phi(l_1,\ldots,l_K) \times \\ &\quad \binom{n_a - 2}{l_a} \prod_{m \neq a} \binom{n_m}{l_m} S_{n_1,\ldots,n_K}(t; l_1,\ldots,l_K). \end{aligned}$$

where $U_3' = \{(l_1,...,l_K) : 0 \leq l_a < n_a - 1, 0 \leq l_m \leq n_m, m \neq a\}$.

It can be shown that $JRI(a,a)$ is again the average of all the joint reliability importance measures $JRI(i,j)$, as presented in Section 4, where i and j represent any two different components of type a.

Figure 9a shows the JRI between two components of type 1 for the system in Figure 1, while Figure 9b shows the JRI between one component of type 1 and one component of type 2. These JRIs are from the black box perspective as discussed in this section, so with only the survival signature assumed to be known for the full system. Hence these JRIs they do not hold for specific components but are average values over all possible pairs of components of the respective types.

7. Concluding remarks

In this paper, we have presented expressions for the marginal and joint reliability measures for a coherent system that consists of multiple types of dependent components. Our method is based on the concept of survival signature which is a useful tool to study systems composed of multiple types of components. The expressions obtained in the present paper generalize and extend the results in Gertsbakh and Shpungin [13], Eryilmaz [8], and Eryilmaz et al. [9]. We have also presented novel importance measures for black box systems for which only the survival signature is available.

Although in the present paper we have studied well-known classical importance measures, the method based on survival signature seems potentially useful to study other importance measures. This will be among our future research problems.

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References

[1] Armstrong, M.J. Joint reliability importance of components, IEEE Transactions on Reliability, 44, 408-412, 1995.

[2] Birnbaum, Z.W. On the importance of different components in a multicomponent system. In Multivariate Analysis, II, Academic Press, New York, 581-592.

[3] Borgonovo, E., Aliee, H., Glaß, M., Teich, J. A new time-independent reliability importance measure, European Journal of Operational Research, 254, 427-442, 2016.

[4] Boushaba, M. and Eryilmaz, S. Joint reliability importance in a binary k-out-of-n:G system with exchangeable dependent components, Quality Technology and Quantitative Management, 11, 453-460, 2014.

[5] Coolen F.P.A., Coolen-Maturi T. Generalizing the signature to systems with multiple types of components. In: Complex systems and dependability. Springer; Berlin, 2012. p. 115–30.

[6] Coolen F.P.A., Coolen-Maturi T. Modelling uncertain aspects of system dependability with survival signatures. In: Dependability problems of complex information systems. Springer; Berlin, 2015. p. 19–34

[7] Eryilmaz, S. Joint reliability importance in linear m-consecutive-k-out-of-n:F systems, IEEE Transactions on Reliability, 62, 862–869, 2013.

[8] Eryilmaz, S. Component importance in coherent systems with exchangeable components, Journal of Applied Probability, 52, 851-863, 2015.

[9] Eryilmaz, S., Oruc, O. and Oger, V. Joint reliability importance in coherent systems with exchangeable dependent components, IEEE Transactions on Reliability, 65, 1562-1570, 2016.

[10] Eryilmaz, S. The concept of weak exchangeability and its applications, Metrika, 80, 259-271, 2017.

[11] Feng, G., Patelli, E., Beer, M. and Coolen, F.P.A. Imprecise system reliability and component importance based on survival signature, Reliability Engineering and System Safety, 150, 116-125, 2016.

[12] Gao, X., Cui, L. and Li, J. Analysis for joint importance of components in a coherent system, European Journal of Operational Research, 182, 282-299, 2007.

[13] Gertsbakh, I.B. and Shpungin, Y. Combinatorial approach to computing component importance indexes in coherent systems, Probability in the Engineering and Informational Sciences, vol. 26, pp. 117-128, 2012.

[14] Hagstrom, J.N. Redundancy, substitutes and complements in system reliability, Technical report, College of Business Administration, University of Illinois, 1990.

[15] Hong, J.S. and Lie, C.H. Joint reliability-importance of two edges in an undirected network, IEEE Transactions on Reliability, 42, 17-33, 1993.

[16] Hong, S., Koo, H.Y. and Lie, C.H. Joint reliability importance of k-out-of-n systems, European Journal of Operational Research, 142, 539-547, 2002.

[17] Kuo, W. and Zhu, X. Importance Measures in Reliability, Risk, and Optimization: Principles and Applications, Wiley, 2012.

[18] Kvassay, M., Zaitseva, E. and Levashenko, V. Importance analysis of multi-state systems based on tools of logical differential calculus, Reliability Engineering and System Safety, 165, 302-316, 2017.

[19] Lu, L. and Jiang, J. Joint failure importance for noncoherent fault trees, IEEE Transactions on Reliability, 56, 435-443, 2007.

[20] Patelli, E., Feng, G., Coolen, F.P.A. and Coolen-Maturi, T. Simulation methods for system reliability using the survival signature, Reliability Engineering and System Safety, 167, 327-337, 2017.

[21] Rani, M. , Jain, K. and Dewan, I. On conditional marginal and conditional joint reliability importance, International Journal of Reliability, Quality and Safety Engineering, 18, 119-138, 2011.

[22] Reed, S. An efficient algorithm for exact computation of system and survival signatures using binary decision diagrams, Reliability Engineering and System Safety, 165, 257-267, 2017.

[23] Wu, S. and Coolen, F.P.A. A cost-based importance measure for system components: An extension of the Birnbaum importance, European Journal of Operational Research, 225, 189-195, 2013.

[24] Zhai, Q., Yang, J., Xie, M., Zhao, Y. Generalized moment-independent importance measures based on Minkowski distance, European Journal of Operational Research, 239, 449-455, 2014.

[25] Zhu, X., Boushaba, M. and Reghioua, M. Joint reliability importance in a consecutive k-out-of-n:F system and an m-consecutive-k-out-of-n:F systems for Markov-dependent components, IEEE Transactions on Reliability, 64, 784–798, 2015.

[26] Zhu, X., Boushaba, M. and Reghioua, M. Reliability and joint reliability importance in a consecutive-k-within-m-out-of-n:F system with Markov dependent components, IEEE Transactions on Reliability, 65, 802–815, 2016.

[27] Zhu, X. and Boushaba, M. linear weighted (n,f,k) system for non-homogeneous Markov-dependent components, IISE Transactions, 49, 722-736, 2017.