New reliability model for complex systems based on stochastic processes and survival signature

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Abstract For systems with complicated structures, reliability analysis based on survival signature has been carried out by modelling time-to-failure data with specific distributions. However, for highly reliable systems, only little or no failure data may be available. To enable reliability analysis without failure data, a new generalised reliability method is proposed for complex systems, based on the survival signature and using stochastic processes to model degradation. The combination of the survival signature and stochastic processes enables the proposed method to be applied to complex systems with different structures and stochastically degrading components. First, system reliability is analysed based on the survival signature and a generalised stochastic process. Then, component reliability analysis based on the generalised stochastic process is introduced using Wiener and Gamma processes. Finally, the approach presented in this paper is illustrated using two numerical examples, and the estimation results are compared with those calculated using failure time distribution functions.

Keywords: reliability; complex systems; degradation; stochastic processes; survival signature.

1. Introduction

Reliability analysis is essential to ensure the proper operation and functionality of systems and infrastructures. The accurate estimation of system reliability enables timely maintenance and reduces the likelihood of severe accidents (Zio, 2016). The reliability of systems with simple structures, such as series, parallel, and series-parallel systems, can be easily calculated after obtaining the reliability of the components (Xiao, Zhang, & Gao, 2020). For systems with more complex structures (Patelli et al., 2017), such as bridge structures (Behrensdorf, Regenhardt, Broggi, & Beer, 2021) or network structures (Da, Chan, & Xu, 2018), which cannot be simplified by alternative series and parallel subsystems, it is

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difficult to analyse the system reliability using the traditional approach. Therefore, a new reliability model for complex systems is proposed which combines the survival signature and degradation processes.

In recent research, system reliability analysis has been widely carried out by modelling degradation data with various stochastic processes, such as the Wiener (Zhang, Si, Hu, & Lei, 2018), Gamma (Wang, Wang, Hong, & Jiang, 2021), and inverse Gaussian processes (Kong, Yang, & Li, 2022). Kong, Yang, & Li (2022) used factor analysis for degradation interactions and proposed a general system reliability model based on diverse stochastic processes. Li, Sadoughi, Hu, & Hu (2020) proposed a hybrid Gaussian reliability model for systems with series and parallel structures by measuring the dependency among the components with a randomized dependence coefficient. Gao, Cui, & Qiu (2019) proposed a system reliability model with competing failure modes, using novel interaction patterns based on the Wiener process. Dong, Cui, & Si (2020) proposed a two-stage system reliability model with two performance characteristics based on bivariate Wiener processes. However, the systems in Gao et al. (2019) and Dong et al. (2020) were considered to function as one component, and the system structures were neglected. Xu, Liang, Li, & Wang (2021) proposed an optimal condition-based maintenance strategy for systems with degradation interactions and imperfect maintenance, based on multiple stochastic processes. Yousefi, Coit, & Song (2020) proposed a new reliability model for series and parallel systems by separating the components into different clusters, with the degradation dependency between the components modelled using a Gamma process. Dong, Liu, Bae, & Cao (2021) proposed a reliability model for series and parallel systems with three different shock damage patterns based on the Gamma process. In the above research on system reliability, the reliability at the system level has been mainly analysed based on stochastic degradation processes. However, the systems are mainly considered to be composed of series and parallel subsystems, or just to function as a single component. To the best of our knowledge, degradation-based reliability analyses for complex systems with complicated structures have not yet been performed. However, many practical systems, such as aerospace and energy supply systems, are designed with complicated bridge and network structures. This paper proposes a new reliability method for such complex systems, based on stochastic degradation processes and the survival signature.

The survival signature, proposed by Coolen & Coolen-Maturi (2013), enables reliability quantification for large systems with multiple types of components by efficient structure analysis. If a system consists of K types of components and m_k components for type k, where $k = 1, 2, \ldots, K$, then there are $2^{(m_1+m_2+\ldots+m_K)}$ entries of the structure function of the system, so computation of reliability for large systems is very challenging. However, the survival signature requires only $\prod_{k=1}^{K}(m_k+1)$ entries making the calculations substantially more efficient, in particular for large systems with relatively few component types. More details about structural analysis using the survival signature are given in **Section**

2.

Computation of the survival signature also requires substantial effort for large systems. Reed (2017) proposed an efficient method for calculating the survival signature for large systems, based on binary decision diagrams. Xu et al. (2019) presented a new method to compute the survival signature using the universal generating function. Behrensdorf, Regenhardt, Broggi, & Beer (2021) developed an efficient numerical method to calculate the survival signature based on percolation theory and Monte Carlo simulation, and the proposed method was applied to large practical systems with hundreds of components. In addition to overcoming the computational limitations, extensive studies have been conducted to extend the application of the survival signatures to various types of systems. Coolen & Coolen-Maturi (2016) combined an imprecise structure function with the survival signature. Their method enables simplification of the system structure and reliability analysis by focusing on a subset of all components. Qin & Coolen (2022) proposed the survival signature for multistate systems with multi-state components. Due to its efficient calculation and extended application, the survival signature has become an important tool for practical reliability analysis of large systems and networks.

Research on reliability of complex systems based on the survival signature has been widely conducted by modelling the component time-to-failure data with probability distributions, such as the normal (Su et al., 2018), Weibull (Walter & Flapper, 2017), and gamma distributions (Salomon, Winnewisser, Wei, Broggi, & Beer, 2021). Based on multiple lifetime distributions, Salomon et al. (2021) studied an imprecise system reliability model by combining the survival signature and fuzzy probability theory. Walter & Flapper (2017) proposed a system reliability evaluation method based on the Weibull distribution and a novel condition-based maintenance strategy, using the survival signature and Bayesian updating. Hashemi, Asadi, & Zarezadeh (2020) assumed several probability distributions for the lifetimes of components and studied corresponding maintenance strategies for complex coherent systems based on the survival signature. The reliability of complex systems with multiple components can be quantified based on the survival signature, the main challenge is the choice of lifetime distributions to model the reliability of the components, as failure time data may be sparse. Indeed, for complex systems with high reliability, such as aerospace systems and nuclear systems, failures are costly and potentially disastrous. Hence, it may not be easy to obtain sufficient failure data of every type of component through life tests, even with the help of accelerating life testing, because few failures or even no failures may occur within a reasonable experimental time (Freitas, de Toledo, Colosimo, & Pires, 2009; Ye, & Xie, 2015). In such cases, to complete the system reliability analysis, a new general reliability model for complex systems is proposed based on the theory of stochastic degradation processes combined with the survival signature.

In contrast to the existing reliability methods for complex systems based on survival signature and time-to-failure data, a new reliability method for complex systems is proposed based on the survival signature in combination with degradation data modelled by stochastic processes. In addition, compared to the existing degradation-based reliability methods for series and parallel systems, the new reliability method is proposed by employing the survival signature, which enables the proposed method to be applied to complex systems consisting of multiple components with general structures, including series, parallel, bridge, and network structures.

This paper is organised as follows. In **Section 2**, a new generalised reliability model for complex systems based on the survival signature and a general stochastic process is proposed. **Section 3** completes the parameter estimation for the general stochastic process based on the expectation-maximization (EM) algorithm. This section also introduces dynamic reliability models for different types of components by modelling the degradation data using the Wiener and Gamma processes. In **Section 4**, the proposed method is illustrated by numerical examples to verify its validity and accuracy. **Section 5** concludes the study and discusses future work.

2. Dynamic system reliability analysis based on the survival signature and stochastic processes

In previous research on system reliability based on the survival signature, the quantification of reliability of components and systems is mainly based on assumed probability distributions for time-to-failure data (Aslett, Coolen, & Wilson, 2015). But as an increasing number of products are designed with highly reliable performance characteristics, it is becoming increasingly difficult to obtain sufficient failure data due to constraints on time and budgets for experiments. For example, components of aeronautical devices and nuclear plants are of crucial importance and they are designed with high reliability to avoid catastrophic accidents, making it difficult to obtain substantial sets of failure data. However, degradation data, which contains large amounts of information on health or status of components before actual failure, could be collected by monitoring components under normal working conditions, or by experiments in which these components are exposed to accelerated stress (Zhang, Si, Hu, & Lei, 2018; Shahraki, Yadav, & Liao, 2017). Therefore, in this section, dynamic system reliability (Liu et al., 2015; Liu & Zio, 2016), defined as the reliability calculated based on stochastic degradation data of components and systems, is considered by modelling component degradation with a general stochastic process, and a new generalised dynamic system reliability model is proposed based on the survival signature and the stochastic processes.

Consider a system composed of K types of components. It is assumed that components of the same type operate under the same operating conditions and follow an identical and independent degradation process. If the components are similar but operate under different operating conditions, such as different stress levels, then the degradation processes of the components are different, and the components should be considered as different types. Let m_k represent the number of components of type k and l_k be the number of working components of type k, where $k = 1, 2, \ldots, K$. Let vector $\boldsymbol{Y} = (\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_k, \ldots, \boldsymbol{Y}_K)$ represent the operational status of the system, where $\boldsymbol{Y}_k = \left(y_{1_k}, y_{2_k}, \cdots, y_{i_k}, \cdots, y_{m_k}\right)$.

$$y_{i_k} = \begin{cases} 1 & \text{if the } i_k \text{th component works} \\ 0 & \text{if the } i_k \text{th component fails} \end{cases} \tag{1}$$

where i_k the index of the component, $i_k = 1_k, 2_k, \ldots, m_k$. The structure function of the system is denoted as $\phi(\boldsymbol{Y})$, such that $\phi(\boldsymbol{Y}) = 1$ denoted that the system functions and $\phi(\boldsymbol{Y}) = 0$ denotes that the system does not function. The survival signature of the system with K types of components can be derived as follows (Eryilmaz, Coolen, & Coolen-Maturi, 2018):

$$\Phi_{\mathrm{S}}\left(l_1, l_2, \cdots, l_K\right) = \left[\prod_{k=1}^{K}\binom{m_k}{l_k}\right]^{-1} \times \sum_{\boldsymbol{Y}\in \boldsymbol{S}_{l_k}} \phi(\boldsymbol{Y}) \tag{2}$$

where $\boldsymbol{S}_{l_k}$ denotes the set of all possible state vectors of the system when the number of working components of the k-th type is l_k.

The reliability of the system can be derived as:

$$R_s(t) = P(T_s > t) = \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left[\Phi_{\mathrm{S}}\left(l_1, l_2, \cdots, l_K\right)\prod_{k=1}^{K} P\left(C_{t,k} = l_k\right)\right] \tag{3}$$

where $P(C_{t,k} = l_k)$ is the probability that the number of working components of the k-th type is l_k at time t, which can be expressed as:

$$P\left(C_{t,k} = l_k\right) = \binom{m_k}{l_k}\left[R_k(t)\right]^{l_k}\left[1 - R_k(t)\right]^{m_k - l_k} \tag{4}$$

where $R_k(t)$ is the reliability of a component of type k.

The system reliability, based on the survival signature in Eq. (3), can be expressed as:

$$R_s(t) = \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left[\Phi_{\mathrm{S}}\left(l_1, l_2, \cdots, l_K\right)\prod_{k=1}^{K}\binom{m_k}{l_k}\left[R_k(t)\right]^{l_k}\left[1 - R_k(t)\right]^{m_k - l_k}\right] \tag{5}$$

To illustrate the system reliability analysis based on the survival signature, the reliability analysis of a multi-component system with a bridge structure is shown in **Appendix A**. In previous system reliability research based on the survival signature, $R_k(t)$ is derived by modelling the time-to-failure data with probability distributions. In this paper, the reliability $R_k(t)$ is calculated by modelling the degradation data with a generalised stochastic process.

Let $X_{i_k}(t)$ denote the degradation value of the i-th component of type k, which is supposed to follow a stochastic process, expressed as $X_{i_k}(t) \sim H\left(t; \boldsymbol{\Theta}_k, \boldsymbol{\Theta}'_{i_k}\right)$, where $H(\cdot)$ is the degradation function and $\boldsymbol{\Theta}_k$ is the matrix of the deterministic parameters representing the common characteristics of the components of type k. The value of $\boldsymbol{\Theta}_k$ is obtained by parameter estimation based on degradation data. $\boldsymbol{\Theta}'_{i_k}$ is the matrix of the random parameters that represents the heterogeneous characteristics of components of type k. The random parameters are assumed to follow known distributions with deterministic parameters. The reliability of the i_k-th component, given $\boldsymbol{\Theta}'_{i_k}$, can be calculated by two methods. One is calculated as the probability that the degradation value of the i_k-th component is less than the failure threshold D_k, which is expressed as:

$$R_{i_k}\left(t\left|\boldsymbol{\Theta}'_{i_k}\right.\right)=P\left(X_{i_k}(t)\le D_k\left|\boldsymbol{\Theta}'_{i_k}\right.\right)=F_{X_{i_k}}\left(D_k\left|\boldsymbol{\Theta}'_{i_k}\right.\right) \tag{6}$$

where D_k is the failure threshold of the k-th type of component, and $F_{X_{i_k}}\left(D_k\left|\boldsymbol{\Theta}'_{i_k}\right.\right)$ is the conditional cumulative distribution function (CDF) of $X_{i_k}(t)$ when $X_{i_k}(t)=D_k$. Alternatively, the reliability of the i_k-th component can also be defined as the probability for the event that the first time the degradation of the i_k-th component reaches the failure threshold is greater than the designed lifetime t. Then, $R_{i_k}\left(t\left|\boldsymbol{\Theta}'_{i_k}\right.\right)$ can be expressed as:

$$R_{i_k}\left(t\left|\boldsymbol{\Theta}'_{i_k}\right.\right)=P\left(T_{i_k}\ge t\left|\boldsymbol{\Theta}'_{i_k}\right.\right)=1-F_{i_k}\left(t\left|\boldsymbol{\Theta}'_{i_k}\right.\right) \tag{7}$$

where T_{i_k} is the first time that the degradation of the i_k-th component reaches the failure threshold D_k and $F_{i_k}\left(t\left|\boldsymbol{\Theta}'_{i_k}\right.\right)$ is the conditional first passage time (FPT) of the i_k-th component.

These two methods enable calculation of $R_{i_k}\left(t\left|\boldsymbol{\Theta}'_{i_k}\right.\right)$. However, the computational complexity of the implementation of these two methods depends on the stochastic processes. For example, if the component degradation follows the Wiener process, the theoretical equation for calculating the reliability based on CDF cannot be easily obtained by Eq. (6), but the FPT is commonly considered to follow the inverse Gaussian process, and the reliability in Eq. (7) can be derived easily. When the degradation follows the Gamma process, the reliability can be approximately calculated by Eq. (7) based on the Birnbaum-Saunders distribution (Pan & Balakrishnan, 2011), but Eq. (6) is more often applied to calculate the reliability theoretically. For other stochastic processes, such as the inverse Gaussian process, more details can be found in Li et al. (2020) and Xu et al. (2021).

After obtaining $R_{i_k}\left(t\left|\boldsymbol{\Theta}'_{i_k}\right.\right)$, the reliability of the i_k-th component can be derived as follows:

$$R_{i_k}(t)=\int_{\boldsymbol{\Theta}'_{i_k}\in\boldsymbol{G}} R_{i_k}\left(t\left|\boldsymbol{\Theta}'_{i_k}\right.\right)f_{\boldsymbol{\Theta}'_{i_k}}\left(\boldsymbol{\Theta}'_{i_k}\right)d\boldsymbol{\Theta}'_{i_k} \tag{8}$$

where $f_{\boldsymbol{\Theta}'_{i_k}}\left(\boldsymbol{\Theta}'_{i_k}\right)$ is the joint probability density function of the random parameters and $\boldsymbol{G}$ is the integration region composed of the range of the random parameters in the matrix $\boldsymbol{\Theta}'_{i_k}$. As shown in Eq. (8), the reliability of the i_k-th component $R_{i_k}(t)$ is calculated by the integral over the random parameters $\boldsymbol{\Theta}'_{i_k}$. Hence, $R_{i_k}(t)$ is a function of t, and it is independent of the random parameters $\boldsymbol{\Theta}'_{i_k}$. Let $R_k(t)$ represent the dynamic reliability of a component of type k, then Eq. (8) can be expressed as:

$$\begin{aligned} R_k(t)&=\int_{\boldsymbol{\Theta}'_{i_k}\in\boldsymbol{G}} R_{i_k}\left(t\left|\boldsymbol{\Theta}'_{i_k}\right.\right)f_{\boldsymbol{\Theta}'_{i_k}}\left(\boldsymbol{\Theta}'_{i_k}\right)d\boldsymbol{\Theta}'_{i_k} \\ &=E_{\boldsymbol{\Theta}'_{i_k}}\left[R_{i_k}\left(t\left|\boldsymbol{\Theta}'_{i_k}\right.\right)\right] \end{aligned} \tag{9}$$

Finally, by substituting Eq. (9) into Eq. (5), a new generalised dynamic reliability model for complex systems with multiple components can be obtained, based on the survival signature and stochastic processes.

$$
\begin{aligned}
R_S(t) &= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left[\Phi_S(l_1,l_2,\cdots,l_K)\prod_{k=1}^{K}\binom{m_k}{l_k}\left[R_k(t)\right]^{l_k}\left[1-R_k(t)\right]^{m_k-l_k}\right] \\
&= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left\{\Phi_S(l_1,l_2,\cdots,l_K)\prod_{k=1}^{K}\binom{m_k}{l_k}\left\{E_{\boldsymbol{\Theta}'_{i_k}}\left[R_{i_k}\left(t\middle|\boldsymbol{\Theta}'_{i_k}\right)\right]\right\}^{l_k}\left\{1-E_{\boldsymbol{\Theta}'_{i_k}}\left[R_{i_k}\left(t\middle|\boldsymbol{\Theta}'_{i_k}\right)\right]\right\}^{m_k-l_k}\right\}
\end{aligned}
\tag{10}
$$

Using the survival signature enables the proposed model to be applied to systems with complex structures (e.g., bridge and network structures), and degradation of components of different types can be modelled by different stochastic processes. **Section 3** presents reliability estimation based on the Wiener and Gamma degradation processes in detail.

3. Dynamic component reliability analysis based on stochastic processes

To apply the proposed dynamic system reliability method, the parameters of the degradation processes of components need to be estimated based on degradation data. In this section, the EM algorithm (Mazzarisi, Barucca, Lillo, & Tantari, 2020), which can deal with censored and truncated data, is introduced for parameter estimation. Then the reliability of each type of component is derived based on the degradation processes of the components.

3.1 Parameter estimation for stochastic processes based on the EM algorithm

Before introducing the EM algorithm, a short overview of the parameter estimation methods is provided. There are three main methods for parameter estimation: traditional methods based on statistics, such as the maximum likelihood estimation (MLE) method (Zhang et al., 2021); methods based on Bayesian theory, such as the EM method (Zhao, Chen, Gaudoin, & Doyen, 2021), the Kalman filtering (KF) method (Si, Wang, Hu, & Zhou, 2011), and the particle filtering (PF) method (Li, Lei, Lin, & Ding, 2015); and methods based on the artificial neural network (ANN), such as the back-propagation (BP) network (Zhou et al., 2021) and the convolutional neural network (CNN) (Vasconcelos, Kijanka, & Urban, 2021). Different methods are suitable for estimating the parameters of models with different characteristics. For models with explicit expressions of likelihood functions and complete data, it is convenient to estimate the parameters by the MLE method. But if there are missing or latent data, then the EM algorithm needs to be utilized. Furthermore, if the data are obtained with measuring errors, then it is better to use stochastic filtering methods such as the KF and PF methods. For models established with big data, the ANN-based methods are more often used.

In this paper, possible measurement errors are neglected, degradation data of components, $X_{i_k}(t)$, are assumed to be available, and the values of random parameters are unknown. Hence, the EM method is selected to estimate the parameters of the degradation process based on possibly incomplete data. The two steps of the EM algorithm can be expressed as follows:

1) E-step: Calculate the conditional expectation of the complete log-likelihood function at iteration q, named the Q-function, which can be described as:

$$
\begin{aligned}
Q\left(\boldsymbol{\Theta}_k \middle| \boldsymbol{\Theta}_k^{(q)}\right) &= E\left[\ln\left(L\left(\boldsymbol{\Theta}_k \middle| \boldsymbol{Z}_k\right)\right) \middle| \boldsymbol{Z}_{k\text{obs}}, \boldsymbol{\Theta}_k^{(q)}\right] \\
&= E\left[\ln\left(f_{\boldsymbol{Z}_k}\left(\boldsymbol{Z}_k \middle| \boldsymbol{\Theta}_k\right)\right) \middle| \boldsymbol{Z}_{k\text{obs}}, \boldsymbol{\Theta}_k^{(q)}\right] \\
&= E\left[\ln\left(f_{\boldsymbol{Z}_{k\text{obs}}|\boldsymbol{Z}_{k\text{miss}}}(\boldsymbol{Z}_{k\text{obs}} \middle| \boldsymbol{Z}_{k\text{miss}}, \boldsymbol{\Theta}_k)\right) \middle| \boldsymbol{Z}_{k\text{obs}}, \boldsymbol{\Theta}_k^{(q)}\right] \\
&\quad + E\left[\ln\left(f_{\boldsymbol{Z}_{k\text{miss}}}(\boldsymbol{Z}_{k\text{miss}} \middle| \boldsymbol{\Theta}_k)\right) \middle| \boldsymbol{Z}_{k\text{obs}}, \boldsymbol{\Theta}_k^{(q)}\right]
\end{aligned}
\tag{11}
$$

where q is the index of iteration, $\boldsymbol{\Theta}_k$ is the matrix of deterministic parameters to estimate, and $\boldsymbol{Z}_k$ represents the complete data, which includes the observed data and missing data, that is, $\boldsymbol{Z}_k = [\boldsymbol{Z}_{k\text{obs}}, \boldsymbol{Z}_{k\text{miss}}]$. The observed dataset is obtained from the measured degradation data, denoted as $\boldsymbol{Z}_{k\textbf{obs}} = \boldsymbol{\Delta X}_k = \left[\boldsymbol{\Delta X}_{1_k}, \boldsymbol{\Delta X}_{2_k}, \cdots, \boldsymbol{\Delta X}_{m_k}\right]^T$, $\boldsymbol{\Delta X}_{i_k} = \left[\Delta X_{i_k 1_k}, \Delta X_{i_k 2_k}, \cdots, \Delta X_{i_k j_k}, \cdots, \Delta X_{i_k n_k}\right]$, $i_k = 1_k, 2_k, \ldots, m_k$. $\Delta X_{i_k j_k} = X_{i_k}\left(t_{j_k}\right) - X_{i_k}\left(t_{j_k - 1}\right)$ is the j_k-th degradation increment of the i_k-th component. The random parameters are unknown and are denoted as $\boldsymbol{Z}_{k\text{miss}} = \boldsymbol{\Theta}'_k = [\boldsymbol{\Theta}'_{1_k}, \boldsymbol{\Theta}'_{2_k}, \cdots, \boldsymbol{\Theta}'_{m_k}]$. For example, if the degradation of the i_k-th component follows the Wiener process, that is, $X_{i_k}\left(t\right) = \theta_{i_k} + \eta_{i_k} t + \sigma_k^2 B\left(t\right)$, where θ_{i_k} and η_{i_k} are random parameters, and they are supposed to follow the normal distributions with deterministic parameters, that is, $\theta_{i_k} \sim N\left(\mu_{\theta_k}, \sigma_{\theta_k}^2\right)$ and $\eta_{i_k} \sim N\left(\mu_{\eta_k}, \sigma_{\eta_k}^2\right)$, where $\mu_{\theta_k}, \mu_{\eta_k}, \sigma_{\theta_k}^2, \sigma_{\eta_k}^2$, and σ_k are deterministic parameters. The random parameter matrix is $\boldsymbol{\Theta}'_{i_k} = \left[\theta_{i_k}, \eta_{i_k}\right]$, and the deterministic parameter matrix is $\boldsymbol{\Theta}_k = \left[\mu_{\theta_k}, \mu_{\eta_k}, \sigma_{\theta_k}^2, \sigma_{\eta_k}^2, \sigma_k^2\right]$.

2) M-Step: Update $\boldsymbol{\Theta}_k^{(q+1)}$ as $\arg\max_{\boldsymbol{\Theta}_k} \partial Q(\boldsymbol{\Theta}_k \mid \boldsymbol{\Theta}_k^{(q)})$. Let $\partial Q(\boldsymbol{\Theta}_k \mid \boldsymbol{\Theta}_k^{(q)}) / \partial \boldsymbol{\Theta}_k = 0$, and the estimation of deterministic parameters for step q+1 step can be obtained. Additional application details are introduced based on the Wiener and Gamma processes in **Section 3.2** and **Section 3.3**.

3.2 Component reliability analysis based on the Wiener process

The exponential Wiener process is often applied to describe nonlinear degradation processes with increasing degradation rates, such as bearing degradation and reduction of light-emitting diode (LED) lighting (Wang, Balakrishnan, & Guo, et al., 2014; Si et al., 2013). More details about the Wiener processes can be found in the review papers of Zhang, Si, Hu, & Lei (2018) and Shahraki, Yadav, & Liao (2017). If the degradation of components of type k is supposed to follow the exponential Wiener process, then the degradation value of the i-th component of type k, $X'_{i_k}\left(t\right)$, can be expressed as:

$$X'_{i_k}(t)=\varphi_{i_k}+\theta'_{i_k}\exp\left(\eta'_{i_k}t+\sigma_k B(t)-\frac{\sigma_k^2}{2}t\right) \tag{12}$$

where φ_{i_k} is a known constant, which is usually considered to be 0; θ'_{i_k} and η'_{i_k} are random parameters representing the heterogeneous characteristics of components of type k; σ_k is a deterministic parameter representing the increasing random error; $B(t)$ is a standard Brownian motion, k = 1, 2, …, K; K is the number of component types; i_k represents the i-th component of type k, $i_k = 1_k, 2_k, \ldots m_k$, where m_k is the number of components of type k.

To facilitate the calculation, the exponential model can be expressed as follows:

$$\begin{aligned} X_{i_k}(t) &= \ln\left[X'_{i_k}(t)-\varphi_{i_k}\right] \\ &= \ln\theta'_{i_k}+\left(\eta'_{i_k}-\frac{\sigma_k^2}{2}\right)t+\sigma_k^2 B(t) \\ &= \theta_{i_k}+\eta_{i_k}t+\sigma_k^2 B(t) \end{aligned} \tag{13}$$

where $\theta_{i_k}=\ln\theta'_{i_k}$ and $\eta_{i_k}=\eta'_{i_k}-\sigma_k^2/2$ .The random parameters θ_{i_k} and η_{i_k} are usually assumed to follow independent Gaussian distributions, $\theta_{i_k}\sim N\left(\mu_{\theta_k},\sigma_{\theta_k}^2\right)$, and $\eta_{i_k}\sim N\left(\mu_{\eta_k},\sigma_{\eta_k}^2\right)$ (Hashemi et al. 2020; Eryilmaz et al., 2018). Hence, for the exponential Wiener process, the matrix of deterministic parameters is $\boldsymbol{\Theta}_k=\boldsymbol{\Theta}_{k\_\mathrm{W}}=\left[\mu_{\theta_k},\mu_{\eta_k},\sigma_{\theta_k}^2,\sigma_{\eta_k}^2,\sigma_k^2\right]$, and the matrix of random parameters is $\boldsymbol{\Theta}'_{i_k}=\left[\theta_{i_k},\eta_{i_k}\right]$. The j_k-th degradation increment of the i_k-th component can be expressed as:

$$\Delta X_{i_k j_k}=X_{i_k}\left(t_{j_k}\right)-X_{i_k}\left(t_{j_k-1}\right) \tag{14}$$

where $X_{i_k}\left(t_{j_k}\right)$ is the degradation value of the i_k-th component at the j_k-th measuring time, j_k is the j-th degradation state of components of type k, $j_k = 1_k, 2_k, \ldots, n_k$, where n_k is the number of degradation states of components of type k, n_k = 1, 2, …, t_{j_k} is the j-th measuring time for components of type k, where components of the same type are assumed to have the same measuring times. The time increments $t_{j_k}-t_{j_k-1}$ are assumed to be constant and expressed as Δt (Si, et al., 2013). The distributions of the degradation increments of the i_k-th component are as follows:

$$\Delta X_{i_k 1_k}\sim N\left(\theta_{i_k}+\eta_{i_k}\Delta t,\sigma_k^2\Delta t\right) \tag{15}$$

$$\Delta X_{i_k j_k}\sim N\left(\eta_{i_k}\Delta t,\sigma_k^2\Delta t\right), j_k=2_k,3_k,\cdots,n_k \tag{16}$$

The conditional probability density function (PDF) of the j_k-th degradation increment is:

$$f_{\Delta X_{i_k j_k}|\boldsymbol{\Theta}'_{i_k}}\left(\Delta X_{i_k j_k}\left|\boldsymbol{\Theta}'_{i_k}\right.\right)=f_{\Delta X_{i_k j_k}|\theta_{i_k},\eta_{i_k}}\left(\Delta X_{i_k j_k}\left|\theta_{i_k},\eta_{i_k}\right.\right)$$

$$=\begin{cases}\dfrac{1}{\sqrt{2\pi\sigma_k^2\Delta t}}\exp\left[-\dfrac{\left(\Delta X_{i_k j_k}-\left(\theta_{i_k}+\eta_{i_k}\Delta t\right)\right)^2}{2\sigma_k^2\Delta t}\right], & \left(j_k=1_k\right)\\ \dfrac{1}{\sqrt{2\pi\sigma_k^2\Delta t}}\exp\left[-\sum_{j_k=2}^{n_k}\dfrac{\left(\Delta X_{i_k j_k}-\eta_{i_k}\Delta t\right)^2}{2\sigma_k^2\Delta t}\right], & \left(j_k=2_k,3_k,\cdots,n_k\right)\end{cases} \tag{17}$$

The joint PDF of the random parameters θ_{i_k} and η_{i_k} is:

$$\begin{aligned}f_{\boldsymbol{\Theta}'_{i_k}}\left(\boldsymbol{\Theta}'_{i_k}\right)&=f_{\theta_{i_k},\eta_{i_k}}\left(\theta_{i_k},\eta_{i_k}\right)\\&=f_{\theta_{i_k}}\left(\theta_{i_k}\right)f_{\eta_{i_k}}\left(\eta_{i_k}\right)\\&=\frac{1}{2\pi\sigma_{\theta k}\sigma_{\eta k}}\exp\left[-\frac{\left(\theta_{i_k}-\mu_{\theta k}\right)^2}{2\sigma_{\theta k}^2}-\frac{\left(\eta_{i_k}-\mu_{\eta k}\right)^2}{2\sigma_{\eta k}^2}\right]\end{aligned} \tag{18}$$

Let D_k denote the failure threshold of components of type k, for the exponential Wiener process, $D_k=\frac{1}{m_k}\sum_{i_k=1}^{m_k}\ln(D'_k-\varphi_{i_k})$, where D'_k is the original value of the failure threshold for components of type k before the logarithmic transformation. When the degradation of the components follows the Wiener process, the PDF and CDF of the FPT follow the inverse Gaussian distribution (Li, Pan, & Chen, 2014; Si, et al., 2013; Wang, Balakrishnan, & Guo, 2014), which can be expressed as:

$$f_{i_k}\left(t\left|\boldsymbol{\Theta}'_{i_k}\right.\right)=f_{i_k}\left(t\left|\theta_{i_k},\eta_{i_k}\right.\right)=\frac{D_k-\theta_{i_k}}{\sqrt{2\pi\sigma_k^2t^3}}\exp\left(-\frac{\left(D_k-\theta_{i_k}-\eta_{i_k}t\right)^2}{2\sigma_k^2t}\right) \tag{19}$$

$$\begin{aligned}F_{i_k}\left(t\left|\boldsymbol{\Theta}'_{i_k}\right.\right)&=F_{i_k}\left(t\left|\theta_{i_k},\eta_{i_k}\right.\right)\\&=P\left(X_{i_k}\left(t\right)\geq D_k\right)=P(T_{i_k}\leq t)\\&=\Phi\left(\frac{-D_k+\theta_{i_k}+\eta_{i_k}t}{\sigma_k\sqrt{t}}\right)+\exp\left(\frac{2\eta_{i_k}\left(D_k-\theta_{i_k}\right)}{\sigma_k^2}\right)\Phi\left(-\frac{D_k-\theta_{i_k}+\eta_{i_k}t}{\sigma_k\sqrt{t}}\right)\end{aligned} \tag{20}$$

The conditional reliability of the i_k-th component can be derived by Eq. (7), leading to

$$\begin{aligned}R_{i_k}\left(t\left|\boldsymbol{\Theta}'_{i_k}\right.\right)&=R_{i_k}\left(t\left|\theta_{i_k},\eta_{i_k}\right.\right)=1-F_{i_k}\left(t\left|\theta_{i_k},\eta_{i_k}\right.\right)\\&=\Phi\left(\frac{D_k-\theta_{i_k}-\eta_{i_k}t}{\sigma_k\sqrt{t}}\right)-\exp\left(\frac{2\eta_{i_k}\left(D_k-\theta_{i_k}\right)}{\sigma_k^2}\right)\Phi\left(-\frac{D_k-\theta_{i_k}+\eta_{i_k}t}{\sigma_k\sqrt{t}}\right)\end{aligned} \tag{21}$$

The unconditional reliability of a component of type k can be derived by Eq. (9), leading to

$$
\begin{aligned}
R_k(t) &= \int_{\boldsymbol{\Theta}'_{i_k} \in \boldsymbol{G}} R_{i_k}\left(t \middle| \boldsymbol{\Theta}'_{i_k}\right) f_{\boldsymbol{\Theta}'_{i_k}}\left(\boldsymbol{\Theta}'_{i_k}\right) d\boldsymbol{\Theta}'_{i_k} \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_{i_k}\left(t \middle| \theta_{i_k}, \eta_{i_k}\right) f_{\theta_{i_k},\eta_{i_k}}\left(\theta_{i_k}, \eta_{i_k}\right) d\theta_{i_k} d\eta_{i_k} \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[\Phi\left(\frac{D_k - \theta_{i_k} - \eta_{i_k} t}{\sigma_k \sqrt{t}} \right) - \exp\left(\frac{2\eta_{i_k}\left(D_k - \theta_{i_k}\right)}{\sigma_k^2} \right) \Phi\left(-\frac{D_k - \theta_{i_k} + \eta_{i_k} t}{\sigma_k \sqrt{t}} \right) \right] \\
&\quad \cdot \frac{1}{2\pi\sigma_{\theta k}\sigma_{\eta k}} \exp\left[-\frac{\left(\theta_{i_k} - \mu_{\theta k}\right)^2}{2\sigma_{\theta k}^2} - \frac{\left(\eta_{i_k} - \mu_{\eta k}\right)^2}{2\sigma_{\eta k}^2} \right] d\theta_{i_k} d\eta_{i_k}
\end{aligned} \tag{22}
$$

Eq. (22) makes clear that it is difficult to obtain the analytical expression of the primitive function of the integrable function. Hence, a numerical integration method is employed to calculate $R_k(t)$ based on a Monte Carlo simulation, which can be derived as:

$$
\begin{aligned}
R_k(t) &= \int_{\boldsymbol{\Theta}'_{i_k} \in \boldsymbol{G}} R_{i_k}\left(t \middle| \boldsymbol{\Theta}'_{i_k}\right) f_{\boldsymbol{\Theta}'_{i_k}}\left(\boldsymbol{\Theta}'_{i_k}\right) d\boldsymbol{\Theta}'_{i_k} \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_{i_k}\left(t \middle| \theta_{i_k}, \eta_{i_k}\right) f_{\theta_{i_k},\eta_{i_k}}\left(\theta_{i_k}, \eta_{i_k}\right) d\theta_{i_k} d\eta_{i_k} \\
&= E_{\left(\theta_{i_k},\eta_{i_k}\right)}\left[R_{i_k}\left(t \middle| \theta_{i_k}, \eta_{i_k}\right) \right] \\
&\approx \frac{1}{N_g N_h} \sum_{g=1}^{N_g} \sum_{h=1}^{N_h} R_{i_k}\left(t \middle| \theta_{i_k\_g}, \eta_{i_k\_h}\right)
\end{aligned} \tag{23}
$$

where, $\theta_{i_k\_g} \sim N\left(\mu_{\theta_k}, \sigma_{\theta_k}^2\right)$, and $\eta_{i_k\_h} \sim N\left(\mu_{\eta_k}, \sigma_{\eta_k}^2\right)$, N_g and N_h are the sampling sizes of θ_{i_k} and η_{i_k}, which are set to 10,000 for the Monte Carlo simulation in this paper. The sampling sizes can also be larger for higher calculation accuracy, but the calculation time could become much longer.

To calculate $R_k(t)$, the deterministic parameters of the stochastic model based on the Wiener process, $\boldsymbol{\Theta}_k = \boldsymbol{\Theta}_{k\_\mathrm{W}} = \left[\mu_{\theta_k}, \mu_{\eta_k}, \sigma_{\theta_k}^2, \sigma_{\eta_k}^2, \sigma_k^2\right]$, need to be obtained. Based on the observed degradation data $X_{i_k}(t)$, the EM algorithm is applied for the parameter estimation. The calculation details are introduced in **Appendix B** and the parameter estimation results are as follows:

$$
\begin{aligned}
\mu_{\theta k}^{(q+1)} &= \frac{1}{m_k}\sum_{i_k=1}^{m_k}\mu'_{\theta i_k} \\
\mu_{\eta k}^{(q+1)} &= \frac{1}{m_k}\sum_{i_k=1}^{m_k}\mu'_{\eta i_k} \\
\sigma_{\theta k}^{2(q+1)} &= \frac{1}{m_k}\sum_{i_k=1}^{m_k}\left[\left(\mu'_{\theta i_k}-\frac{1}{m_k}\sum_{i_k=1}^{m_k}\mu'_{\theta i_k}\right)^2+\sigma'^2_{\theta i_k}\right] \\
\sigma_{\eta k}^{2(q+1)} &= \frac{1}{m_k}\sum_{i_k=1}^{m_k}\left[\left(\mu'_{\eta i_k}-\frac{1}{m_k}\sum_{i_k=1}^{m_k}\mu'_{\eta i_k}\right)^2+\sigma'^2_{\eta i_k}\right] \\
\sigma_{k}^{2(q+1)} &= \frac{1}{m_k n_k \Delta t}\left(\sum_{i_k=1}^{m_k}\sum_{j_k=1}^{n_k}\Delta X_{i_k j_k}^2-2\sum_{i_k=1}^{m_k}\left(\Delta X_{i_k 1_k}\right)\mu'_{\theta i_k}-2\Delta t\sum_{i_k=1}^{m_k}\left(\mu'_{\eta i_k}\sum_{j_k=1}^{n_k}\Delta X_{i_k j_k}\right)+\sum_{i_k=1}^{m_k}\left(\mu'^2_{\theta i_k}+\sigma'^2_{\theta i_k}\right)\right. \\
&\quad \left.+2\Delta t\sum_{i_k=1}^{m_k}\left(\rho_{i_k}\sigma'_{\theta i_k}\sigma'_{\eta i_k}+\mu'_{\theta i_k}\mu'_{\eta i_k}\right)+n_k\Delta t^2\sum_{i_k=1}^{m_k}\left(\mu'^2_{\eta i_k}+\sigma'^2_{\eta i_k}\right)\right)
\end{aligned}
\tag{24}
$$

The **M-step** is repeated until the difference between the last two values of the Q-function is less than 10^{-6}, that is, $\left|Q\left(\boldsymbol{\Theta}_{k\_\mathrm{W}}\left|\boldsymbol{\Theta}_{k\_\mathrm{W}}^{(q+1)}\right.\right)-Q\left(\boldsymbol{\Theta}_{k\_\mathrm{W}}\left|\boldsymbol{\Theta}_{k\_\mathrm{W}}^{(q)}\right.\right)\right|<10^{-6}$. Then, the parameters converge, and the values of the deterministic parameters, $\boldsymbol{\Theta}_{k\_\mathrm{W}}^{(q+1)}=\left[\mu_{\theta_k}^{(q+1)},\mu_{\eta_k}^{(q+1)},\sigma_{\theta_k}^{(q+1)2},\sigma_{\eta_k}^{(q+1)2},\sigma_{k}^{(q+1)2}\right]$, can be obtained by Eq. (24). Finally, the reliability of components of type k, based on the Wiener process, can be obtained by substituting the estimated parameters into Eq. (23).

3.3 Component reliability analysis based on the Gamma process

The Gamma process is suitable for describing monotonic degradation processes such as abrasion and erosion (Yousefi et al., 2020; Dong et al., 2021). More details about Gamma processes can be found in the reviews by Si, Wang, Hu, & Zhou (2011) and Ye & Xie (2015). If the degradation of components of type k follows a Gamma process, that is, $X_{i_k}(t)\sim Ga\left(t;\alpha_k,\beta_{i_k}\right)$, where $X_{i_k}(0)=0$, α_k is the deterministic shape parameter and β_{i_k} is the random scale parameter of this process for components of type k, with the latter representing the heterogeneous characteristics of components of type k. The shape parameter is assumed to be a linear function of time, that is, $\alpha_k = a_k t$, where a_k is a deterministic parameter. To describe the randomness among the components, the scale parameters for components of type k are assumed to be independent and identically gamma-distributed, that is, $\beta_{i_k}\sim Ga\left(\delta_k,\lambda_k\right)$. Hence, for the Gamma process, the matrix of deterministic parameters is $\boldsymbol{\Theta}_k=\boldsymbol{\Theta}_{k\_\mathrm{G}}=\left[a_k,\delta_k,\lambda_k\right]$, and the matrix of the random parameter matrix is $\boldsymbol{\Theta}'_{i_k}=\left[\beta_{i_k}\right]$. Then, the j_k-th degradation increment for the i_k-th component can be expressed as:

$$
\Delta X_{i_k j_k}=X_{i_k}\left(t_{j_k}\right)-X_{i_k}\left(t_{j_k-1}\right) \tag{25}
$$

$$\Delta X_{i_k j_k} \sim Ga\left(a_k\left(t_{j_k} - t_{j_k-1}\right), \beta_{i_k}\right) \tag{26}$$

where i_k and j_k are the same as those of the Wiener model described in **Section 3.2**.

The conditional PDF of degradation increments, $\Delta X_{i_k j_k}$, is:

$$f_{\Delta X_{i_k j_k}|\boldsymbol{\Theta}'_{i_k}}\left(\Delta X_{i_k j_k} \middle| \boldsymbol{\Theta}'_{i_k}\right) = f_{\Delta X_{i_k j_k}|\beta_{i_k}}\left(\Delta X_{i_k j_k} \middle| \beta_{i_k}\right) = \frac{\Delta X_{i_k j_k}^{\left(a_k \Delta t - 1\right)}}{\Gamma\left(a_k \Delta t\right)} \cdot \beta_{i_k}^{a_k \Delta t} \cdot \exp\left(-\beta_{i_k} \cdot \Delta X_{i_k j_k}\right) \tag{27}$$

where the $\Gamma(\cdot)$ is the gamma function, $\Gamma\left(x_1, x_2\right) = \int_{x_2}^{\infty} t^{x_1 - 1} \exp\left(-t\right) dt$.

The PDF of the random scale parameter, β_{i_k}, is:

$$f_{\boldsymbol{\Theta}'_{i_k}}\left(\boldsymbol{\Theta}'_{i_k}\right) = f_{\beta_{i_k}}\left(\beta_{i_k}\right) = \frac{\beta_{i_k}^{\delta_k - 1}}{\Gamma\left(\delta_k\right)} \cdot \lambda_k^{\delta_k} \cdot \exp\left(-\lambda_k \cdot \beta_{i_k}\right) \tag{28}$$

The conditional reliability of the i_k-th component, calculated by Eq. (6) is:

$$\begin{aligned} R_{i_k}\left(t \middle| \boldsymbol{\Theta}'_{i_k}\right) &= R_{i_k}\left(t \middle| \beta_{i_k}\right) \\ &= P(T_{i_k} \geq t) = P\left(X_{i_k}\left(t\right) \leq D_k\right) \\ &= \frac{1}{\Gamma\left(a_k t\right)} \gamma\left(a_k t, \beta_{i_k} \cdot D_k\right) \end{aligned} \tag{29}$$

where $\gamma(\cdot)$ is the lower incomplete gamma function, $\gamma\left(x_1, x_2\right) = \int_0^{x_2} t^{x_1 - 1} \exp\left(-t\right) dt$.

The reliability of a component of type k, as given by Eq. (9) and based on the Gamma process, can be derived as:

$$\begin{aligned} R_k\left(t\right) &= \int_{\boldsymbol{\Theta}'_{i_k} \in \boldsymbol{G}} R_{i_k}\left(t \middle| \boldsymbol{\Theta}'_{i_k}\right) f_{\boldsymbol{\Theta}'_{i_k}}\left(\boldsymbol{\Theta}'_{i_k}\right) d\boldsymbol{\Theta}'_{i_k} \\ &= \int_{-\infty}^{+\infty} R_{i_k}\left(t \middle| \beta_{i_k}\right) f_{\beta_{i_k}}\left(\beta_{i_k}\right) d\beta_{i_k} \\ &= \int_{-\infty}^{+\infty} \frac{1}{\Gamma\left(a_k t\right)} \gamma\left(a_k t, \beta_{i_k} \cdot D_k\right) \cdot \frac{\beta_{i_k}^{\delta_k - 1}}{\Gamma\left(\delta_k\right)} \cdot \lambda_k^{\delta_k} \cdot \exp\left(-\lambda_k \cdot \beta_{i_k}\right) d\beta_{i_k} \end{aligned} \tag{30}$$

Similar to the calculation of $R_k(t)$ based on the Wiener process, as described in **Section 3.2**, the reliability $R_k(t)$ in Eq. (30) can be calculated using the Monte Carlo method, which can be derived as:

$$\begin{aligned} R_k\left(t\right) &= \int_{\boldsymbol{\Theta}'_{i_k} \in \boldsymbol{G}} R_{i_k}\left(t \middle| \boldsymbol{\Theta}'_{i_k}\right) f_{\boldsymbol{\Theta}'_{i_k}}\left(\boldsymbol{\Theta}'_{i_k}\right) d\boldsymbol{\Theta}'_{i_k} \\ &= \int_{-\infty}^{+\infty} R_{i_k}\left(t \middle| \beta_{i_k}\right) f_{\beta_{i_k}}\left(\beta_{i_k}\right) d\beta_{i_k} \\ &= E_{\beta_{i_k}}\left[R_{i_k}\left(t \middle| \beta_{i_k}\right)\right] \\ &\approx \frac{1}{N_r} \sum_{r=1}^{N_r} R_{i_k}\left(t \middle| \beta_{i_k\_r}\right) \end{aligned} \tag{31}$$

where $\beta_{i_k\_r} \sim Ga\left(\delta_k, \lambda_k\right)$ and N_r is the sampling size of β_{i_k}. As before, the sampling size in the Monte Carlo simulation is set at 10,000.

To calculate the reliability of the k-th type of components $R_k(t)$, the deterministic parameters

$\boldsymbol{\Theta}_k = \boldsymbol{\Theta}_{k\_\mathrm{G}} = \left[a_k, \delta_k, \lambda_k\right]$ need to be estimated. The EM method is used to complete the parameter estimation based on the observed degradation data $X_{i_k}(t)$. The calculation details are provided in **Appendix C**, and the parameter estimation results are as follows:

$$\begin{aligned}
&\Psi\left(a_k^{(q+1)}\Delta t\right) = \frac{1}{m_k n_k}\sum_{i_k=1}^{m_k}\sum_{j_k=1}^{n_k}\ln \Delta X_{i_k j_k}(t) + \Psi\left(n a_k^{(q)}\Delta t + \delta_k^{(q)}\right) - \frac{1}{m_k}\sum_{i_k=1}^{m_k}\ln\left(\sum_{j_k=1}^{n_k}\Delta X_{i_k j_k}(t) + \lambda_k^{(q)}\right)\\
&\lambda_k^{(q+1)} = \frac{m_k \delta_k^{(q+1)}}{\sum_{i_k=1}^{m_k}\dfrac{n_k a_k^{(q)}\Delta t + \delta_k^{(q)}}{\sum_{j_k=1}^{n_k}\Delta X_{i_k j_k}(t) + \lambda_k^{(q)}}}\\
&\Psi\left(\delta_k^{(q+1)}\right) - \ln\left(\delta_k^{(q+1)}\right) = \Psi\left(n_k a_k^{(q)}\Delta t + \delta_k^{(q)}\right) - \frac{1}{m_k}\sum_{i_k=1}^{m_k}\ln\left(\sum_{j_k=1}^{n_k}\Delta X_{i_k j_k}(t) + \lambda_k^{(q)}\right)\\
&\qquad - \ln\frac{1}{m_k}\sum_{i_k=1}^{m_k}\frac{n_k a_k^{(q)}\Delta t + \delta_k^{(q)}}{\sum_{j_k=1}^{n_k}\Delta X_{i_k j_k}(t) + \lambda_k^{(q)}}
\end{aligned} \tag{32}$$

The updating process in Eq. (32) is repeated until the difference between the last two values of the Q-function is less than 10^{-6}, that is, $\left|Q\left(\boldsymbol{\Theta}_{k\_\mathrm{G}} \middle| \boldsymbol{\Theta}_{k\_\mathrm{G}}^{(q+1)}\right) - Q\left(\boldsymbol{\Theta}_{k\_\mathrm{G}} \middle| \boldsymbol{\Theta}_{k\_\mathrm{G}}^{(q)}\right)\right| < 10^{-6}$. Then, the parameters converge, and the values of the deterministic parameters $\boldsymbol{\Theta}_{k\_\mathrm{G}}^{(q)} = \left[a_k^{(q+1)}, \delta_k^{(q+1)}, \lambda_k^{(q+1)}\right]$ can be obtained by Eq. (32). Finally, the reliability of the k-th type of component based on the Gamma process can be obtained by substituting the parameter estimates into Eq. (31).

4. Numerical Examples

In this section, the method proposed in this paper is illustrated by two numerical examples. In the first example, the dynamic reliability is analyzed at the component level and compared with the results of the existing method based on the widely used Weibull distribution. In the second example, the dynamic reliability at the complex system level is analysed and compared with the results of the existing methods based on different distributions.

4.1 Example 1: dynamic reliability analysis at the component level.

In this example, the proposed and existing methods are compared in terms of three aspects: reliability analysis, life prediction, and degradation path modelling. The parameters for the simulation are shown in **Table 1**, which are estimated by the proposed method with the experimental data obtained from the work of Meeker and Escobar (1998). Two sets of data are simulated based on the parameters in **Table 1** and the results are shown in **Figs. 1-2.** Because only a few components failed in the experiments and the failure data are not sufficient to complete the comparison. The first set is the fatigue crack growth data, and the second set is the operating current growth data of lasers. The components in **Fig. 1** fail when the crack size reaches 1.6 inches (Wang, Balakrishnan, & Guo, et al., 2014). Lasers in **Fig. 2** fail when the increase in operating current reaches 10% (Tsai et al., 2012). Note that the time units of the two types of components are different in this section, which does not affect the reliability analysis at the component level. But for reliability analysis at the system level, as shown in **Section 4.2**, it is better to use the same time unit for types of components in one system.

Table 1 The parameters for sampling

	The parameters of the Wiener process				The parameters of the Gamma process			
Name	$\mu_{\theta 1}$	$\mu_{\eta 1}$	$\sigma^2_{\theta 1}$	$\sigma^2_{\eta 1}$	σ^2_1	a_2	λ_2	δ_2
Value	-0.0068	4.21×10^{-6}	2.63×10^{-5}	6.53×10^{-20}	3.40×10^{-9}	0.0374	29.73	1.74

Note: the subscripts 1 and 2 are the indices of types of components

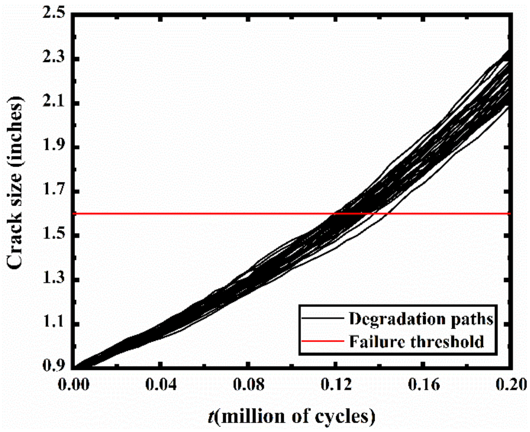


Fig. 1 The fatigue crack growth

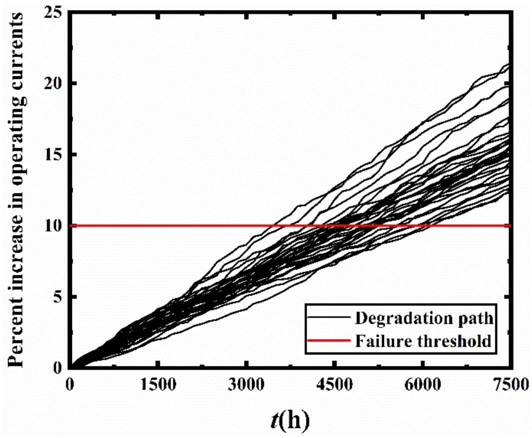


Fig. 2 The percent increase in operating currents

The fatigue crack develops with a nonlinear degradation rate, hence the data in **Fig. 1** are analysed with the exponential Wiener process. The operating currents increase steadily and monotonously, hence the degradation data in **Fig. 2** are modelled with the Gamma process. The parameter estimation processes are omitted here, and the results are shown in Appendix D. After 247 iterations, the estimates for the five deterministic parameters of the Wiener process calculated by Eq. (24) have converged, with the difference between the last two values of the Q-function less than 10^{-6}, that is, $\left|Q\left(\boldsymbol{\Theta}_{1\_\mathrm{W}}\left|\boldsymbol{\Theta}_{1\_\mathrm{W}}^{(q+1)}\right.\right)-Q\left(\boldsymbol{\Theta}_{1\_\mathrm{W}}\left|\boldsymbol{\Theta}_{1\_\mathrm{W}}^{(q)}\right.\right)\right|<10^{-6}$. After 230 iterations, the estimates for the three parameters of the Gamma process calculated by Eq. (32) have converged, as $\left|Q\left(\boldsymbol{\Theta}_{2\_\mathrm{G}}\left|\boldsymbol{\Theta}_{2\_\mathrm{G}}^{(q+1)}\right.\right)-Q\left(\boldsymbol{\Theta}_{2\_\mathrm{G}}\left|\boldsymbol{\Theta}_{2\_\mathrm{G}}^{(q)}\right.\right)\right|<10^{-6}$.

The prediction accuracy of the mean degradation value of a type of component is an important index to check the validity of the proposed method (Wang, Balakrishnan, & Guo, et al., 2014). In **Figs. 3-4**, the solid blue lines represent the means of the predicted degradation paths, which are calculated by Eqs.

(12-14) and Eqs. (25-26) with the parameter estimates given in **Appendix D**. The dashed purple lines represent the means of the empirical degradation paths (the grey background lines). It can be seen from **Figs. 3-4** that the proposed models based on stochastic processes seem to predict the empirical degradation paths well.

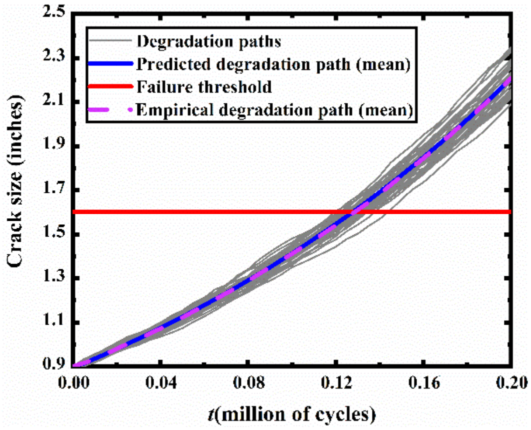


Fig. 3 Comparison of the predicted and empirical fatigue crack growth paths

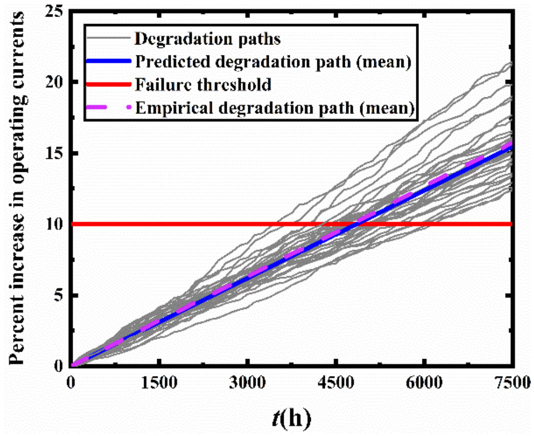


Fig. 4 Comparison of the predicted and empirical operating currents growth paths

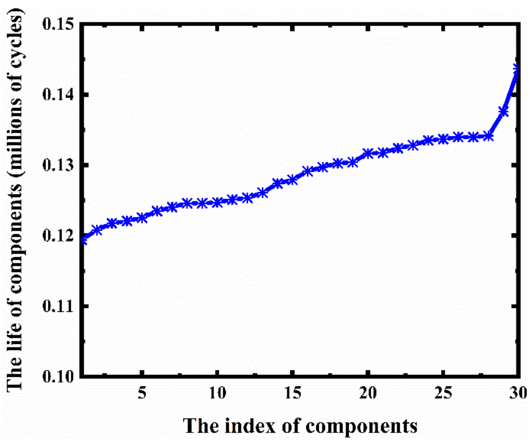


Fig. 5 The lifetimes of components with growing fatigue crack

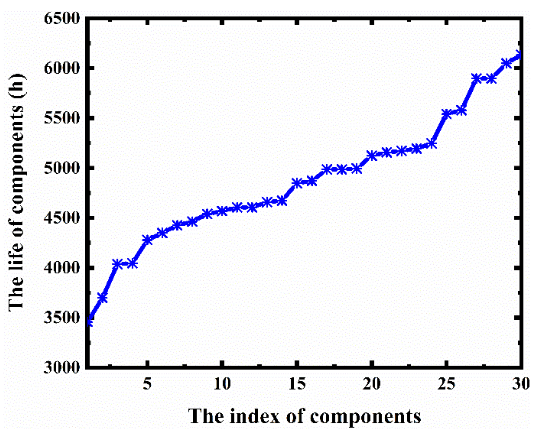


Fig. 6 The lifetimes of components with increasing operating currents

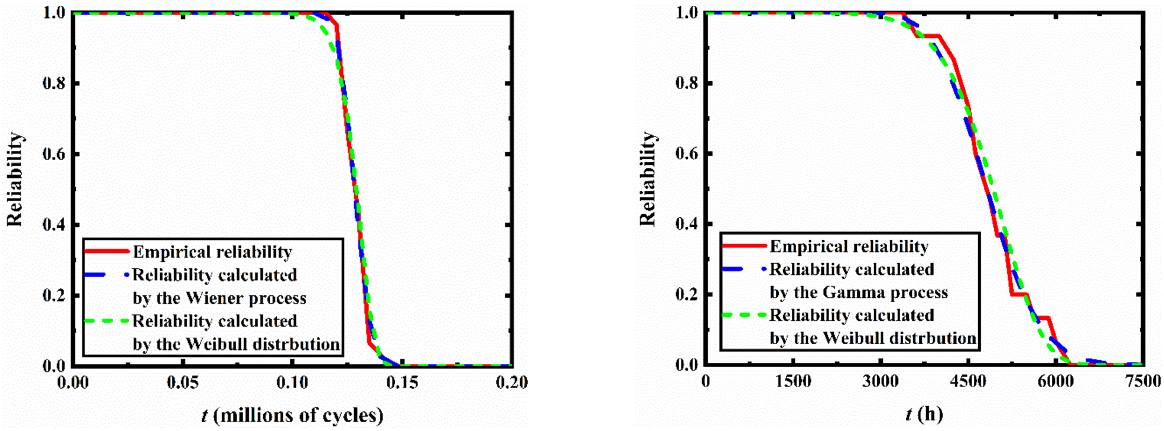
To compare the reliability calculated by the proposed method based on stochastic processes, with the existing method based on the Weibull distribution, firstly, the failure time of the components needs to be obtained by the intersecting point coordinates of the black and red lines shown in **Figs. 1-2**. The results are shown in **Figs. 5-6**. Then, the commonly used Weibull distribution, given in Eq. (33), is employed to model the PDF of the failure time. The MLE (maximum likelihood estimation) method is used to estimate the parameters, and the results are shown in **Table 2.**

$$f\left(t;u_k,v_k\right)=\frac{v_k}{u_k}\left(\frac{t}{u_k}\right)^{v_k-1}\exp\left[-(\frac{t}{u_k})^{v_k}\right],\qquad t\geq 0 \tag{33}$$

where k is the type of component, in this section k = 1, 2.

Table 2 The parameter estimation results of the Weibull distribution

The fatigue crack data		The operating currents data	
u_1	v_1	u_2	v_2
131347.00	22.17	5155.24	8.12



(a) Fatigue crack data **(b)** Operating current data

Fig. 7 The comparison of the empirical and predicted reliabilities of the components

Table 3 The MREs and the MAEs between the predicted and the empirical reliabilities

	Wiener process	Weibull distribution	Gamma process	Weibull distribution
MRE	2.34%	2.82%	4.66%	7.74%
MAE	0.0050	0.0060	0.0098	0.0127

The reliability evaluation results of the proposed stochastic processes and Weibull distribution are compared with the empirical reliability, and the results are shown in **Fig. 7** and **Table 3**. In **Fig. 7**, the solid red lines represent the empirical reliabilities, which are calculated as the ratio of the number of reliable components to the total number of components. The dot-dashed blue lines in **Fig. 7** are the reliabilities calculated by Eqs. (23) and (31), based on the stochastic processes. The dashed green lines in **Fig. 7** represent the reliabilities based on the Weibull distribution. As shown in **Fig. 7**, both the estimated reliabilities based on the stochastic processes and the Weibull distribution fit well with the empirical reliabilities. However, as shown in **Table 3**, both the mean relative errors (MREs) and mean absolute errors (MAEs) of the reliabilities predicted by the proposed stochastic processes are smaller than those predicted by the Weibull distribution, and the MREs of the reliabilities predicted by the proposed stochastic processes are less than 5%. Therefore, the calculation results indicate that the reliabilities of the two types of components can be evaluated more accurately by the proposed stochastic processes than by the Weibull distribution.

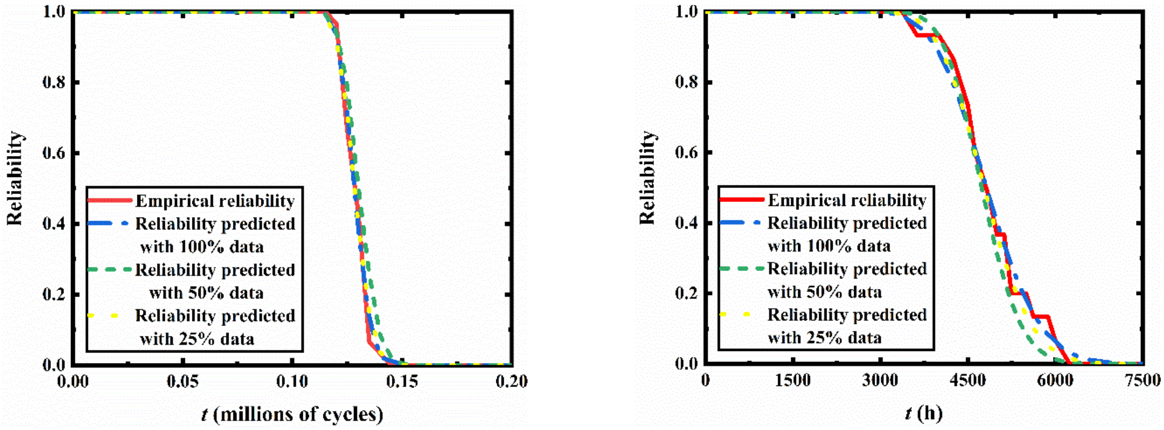
Table 4 Comparison of the empirical and predicted mean lifetime

	Mean lifetime	Relative error
Empirical	0.1284 (millions of cycles)	-
Wiener process	0.1276 (millions of cycles)	-0.62%
Weibull distribution	0.1282 (millions of cycles)	-0.16%
Empirical	4780.57 (h)	-
Gamma process	4843.20 (h)	1.31%
Weibull distribution	4858.54 (h)	1.63%

Table 4 shows the comparison of the mean lifetimes predicted by the stochastic processes and the Weibull distribution, where the empirical mean lifetimes are obtained by the intersection point coordinates of the dashed purple lines and the solid red lines in **Figs. 3-4**, the mean lifetimes predicted by the stochastic processes are obtained by the intersection point coordinates of the solid blue lines and the solid red lines in **Figs. 3-4**, and the mean lifetimes predicted by the Weibull distribution are equal to $E(t)=u_k\Gamma(1+1/v_k)$, where k is the type of the components, k = 1, 2, and $\Gamma(\cdot)$ is the gamma function. The relative errors are also shown in **Table 4**, and the results indicate that the lifetimes of the components can be accurately predicted by both models, so either based on the proposed stochastic processes or on the Weibull distribution.

However, the Weibull distribution is usually applied to evaluate the component reliability and lifetime by using sufficient failure data, which may need to be obtained by destructive experiments. As shown in **Fig. 7**, **Table 3**, and **Table 4**, compared with the existing method, the proposed models can complete the reliability analysis and life prediction at the same, or even higher accuracy level, by using the degradation data, which can be obtained by non-destructive monitoring.

To better address the advantages of the proposed model based on stochastic processes, reliability analysis, life prediction, and degradation modelling are conducted by using fewer degradation data. Compared with the method based on the Weibull distribution, failure data are not necessarily required, and the results in **Figs. 8-9** and **Table 5** show that the reliabilities, lifetimes, and degradation paths can still be accurately predicted with only 50% or 25% data (the data obtained from the start to 50% or 25% of the experimental time). Therefore, the time and budget required for experiments, to enable reliability analysis, life prediction, and degradation modelling can be substantially reduced using stochastic processes instead of probability distributions for the lifetimes.

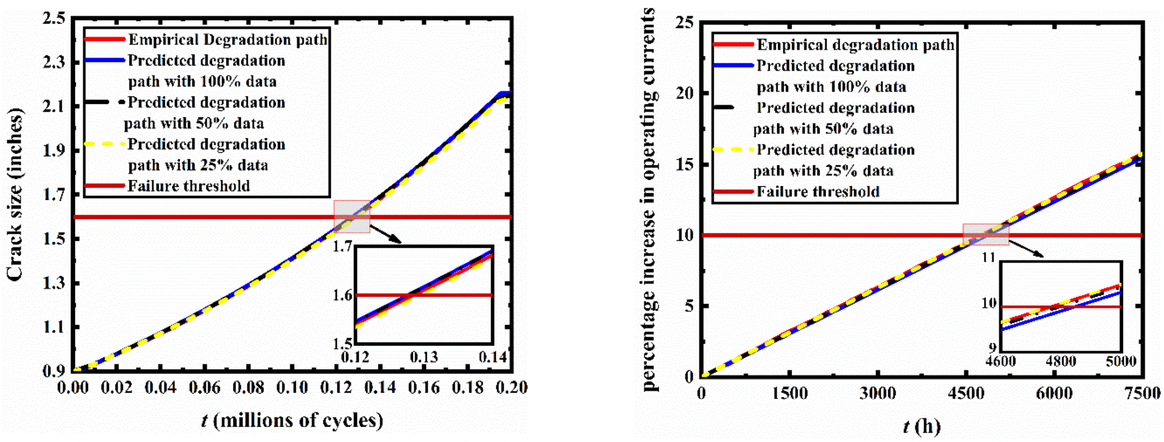


(a) Fatigue crack data **(b)** Operating current data

Fig. 8 Comparison of reliabilities based on different amounts of degradation data

Table 5 Comparison of mean predicted lifetime based on different amounts of degradation data

	Mean lifetime - Wiener process (millions of cycles)	Relative error	Mean lifetime - Gamma process (h)	Relative error
Empirical	0.1284	-	4780.57	-
Predicted with 100% data	0.1276	0.62%	4843.20	1.31%
Predicted with 50% data	0.1278	0.47%	4812.84	0.68%
Predicted with 25% data	0.1296	0.93%	4788.24	0.16%

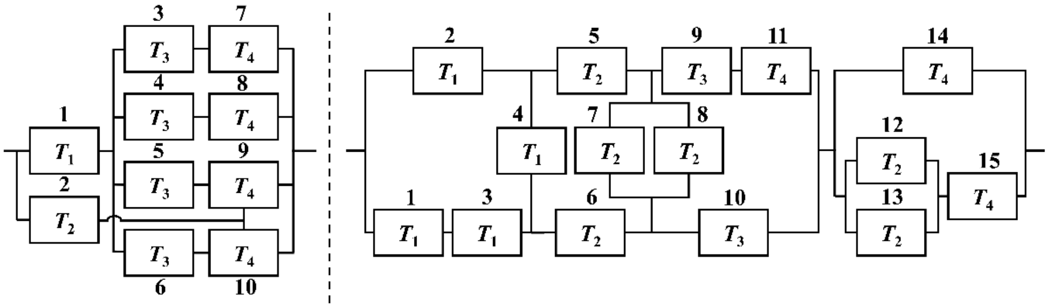


(a) Fatigue crack data **(b)** Operating current data

Fig. 9 Comparison of the empirical and predicted degradation path (mean)

4.2 Example 2: dynamic reliability analysis at the system level

To address the advantages of the proposed method, the system reliability estimated by modelling the degradation data with the proposed stochastic processes is compared with that obtained by analysing the failure data with several commonly used distributions. The system reliabilities calculated with different amounts of degradation data are compared to illustrate the possible reduction of time needed for experiments.

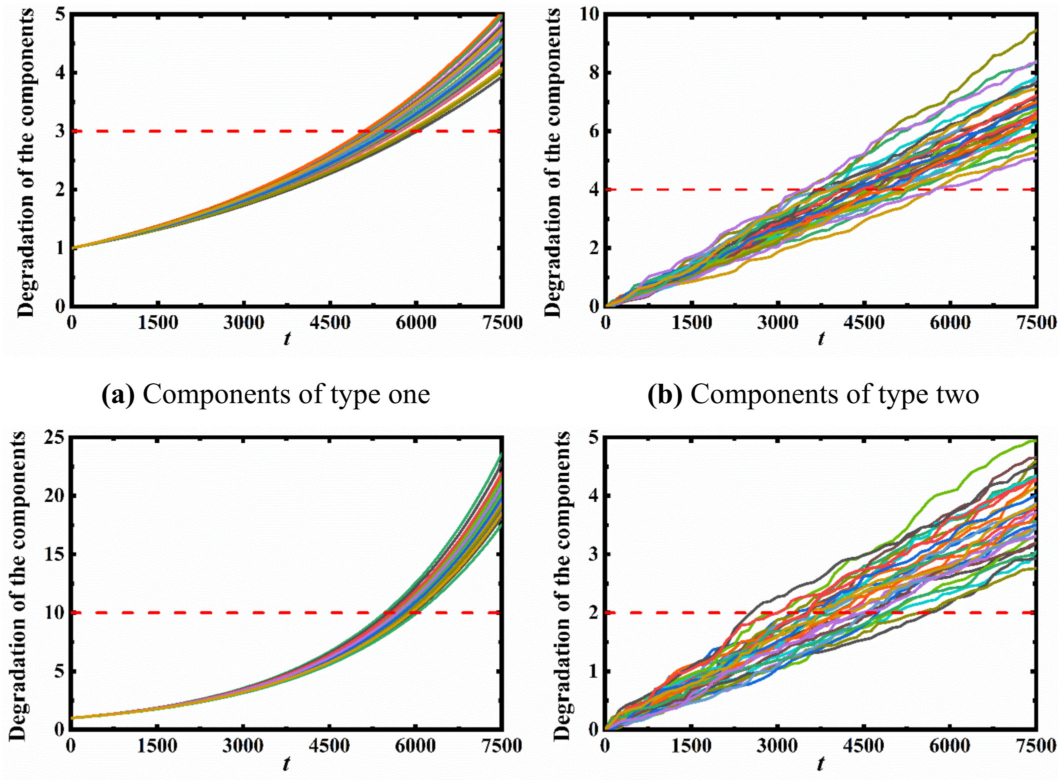


(a) System one: an automotive braking system. (Tavangar, & Hashemi, 2022) **(b)** System two: a system with 15 components of four types. (Huang et al., 2019)

Fig. 10 The reliability block diagrams of complex systems with multiple components of four types.

In this example, two complex systems are considered to illustrate the application of the proposed method. As shown in **Fig. 10**, the first system is an automotive braking system with 10 components. The second system consists of 15 components. Both systems are composed of four types of components. The degradation data and failure threshold of each type of components are shown in **Fig. 11**.

As shown in **Table 6**, the parameters of the four types of components are obtained by analysing the degradation data in **Fig. 11** with Eqs. (24) and (32). Then, by substituting the parameters in **Table 6** into Eqs. (23) and (31), the reliability of each component type can be calculated. To compare the system reliability calculated by the proposed and existing methods, the sets of time-to-failure data for applying the commonly used distributions are obtained by the intersecting point coordinates of the solid and dashed lines in **Fig. 11**, and the results are shown in **Fig. 12**. Based on the time-to-failure data, the parameters of three types of distributions are estimated by applying the MLE method and the results are shown in **Table 7**. By substituting these parameter estimates into the reliability equations in **Table 7**, the reliabilities of the four types of components based on the lifetime distributions are estimated.



(a) Components of type one **(b)** Components of type two

(c) Components of type three **(d)** Components of type four

(Solid lines: degradation paths; dashed lines: failure thresholds)

Fig. 11 The degradation paths of four types of components

Table 6 The parameter estimation results

Component type	Stochastic processes	Values of parameters
1	Wiener	$\boldsymbol{\Theta}_{1\_W}$ = [2.26e-4, 2.01e-4, 2.78e-9, 6.82e-11, 9.77e-11]
2	Gamma	$\boldsymbol{\Theta}_{2\_G}$ = [68.79, 0.03, 1.99]
3	Wiener	$\boldsymbol{\Theta}_{3\_W}$ = [3.63e-4, 4.01e-4, 6.81e-15, 9.55e-11, 2.38e-10]
4	Gamma	$\boldsymbol{\Theta}_{4\_G}$ = [65.58, 0.02, 1.68]

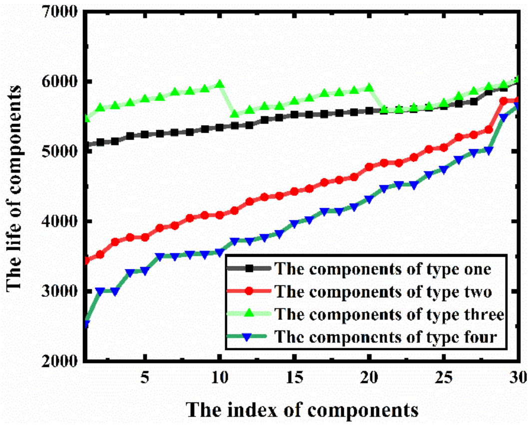


Fig. 12 The time-to-failure data of the four types of components

Table 7 The estimated parameters of different distributions based on the time-to-failure data

Distribution	$R_k(t)$	Component type	Values of parameters
Weibull distribution	$R_{k\_\mathrm{Norm}}(t)=1-\Phi\left(\dfrac{t-\upsilon_k}{\omega_k}\right)$	1	$[u_1, v_1] = [5592.03, 24.14]$
		2	$[u_2, v_2] = [4759.57, 7.92]$
		3	$[u_3, v_3] = [5813.31, 44.43]$
		4	$[u_4, v_4] = [4365.31, 5.95]$
Gamma distribution	$R_{k\_\mathrm{Gam}}(t)=1-\dfrac{1}{\Gamma(\varepsilon_k)}\gamma(\varepsilon_k,\xi_k t)$	1	$[\varepsilon_1, \xi_1] = [585.95, 9.35]$
		2	$[\varepsilon_2, \xi_2] = [55.38, 81.08]$
		3	$[\varepsilon_3, \xi_3] = [585.95, 9.35]$
		4	$[\varepsilon_4, \xi_4] = [30.25, 134.07]$
Normal distribution	$R_{k\_\mathrm{Weib}}(t)=\exp\left(-\left(\dfrac{t}{u_k}\right)^{v_k}\right)$	1	$[\upsilon_1, \omega_1] = [5480.72, 230.96]$
		2	$[\upsilon_2, \omega_2] = [4490.82, 615.31]$
		3	$[\upsilon_3, \omega_3] = [585.95, 9.35]$
		4	$[\upsilon_4, \omega_4] = [4056.08, 746.50]$

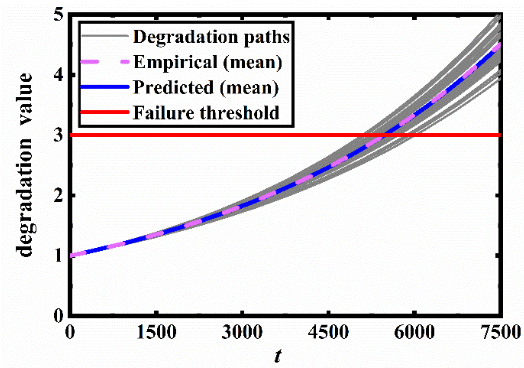
Note: ω_k, υ_k, u_k, v_k, and ε_k are the parameters of the lifetime distribution for components of type k, $\Phi(\cdot)$ is the standard normal cumulative distribution function.

To calculate the reliability of the systems, the survival signatures of the systems are calculated by Eq. (2), and the results are shown in **Table 8**. For the systems in **Fig. 10**, there are respectively 2×2×5×5=100 and 5×7×3×4=420 entries for the survival signatures, hence, only parts of the results are shown in **Table 8**. Then, substituting the reliabilities of the four types of components calculated by Eqs. (23) and (31) and the survival signature results in **Table 8** into Eq. (10), the reliabilities of the two complex systems are obtained by the proposed method, and the results are shown in **Fig. 14**.

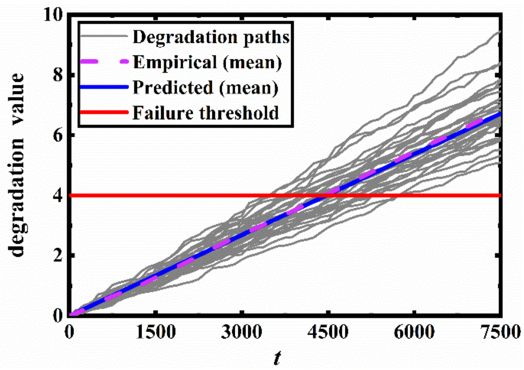
Table 8 The survival signature of the two systems

System one					System two				
l_1	l_2	l_3	l_4	$\Phi_S(l_1, l_2, l_3, l_4)$	l_1	l_2	l_3	l_4	$\Phi_S(l_1, l_2, l_3, l_4)$
0	1	0	1	0.5	1	1	1	2	0.0069
0	1	0	2	0.83	1	1	1	3	0.0208
0	1	0	3	1	1	1	2	2	0.0139
0	1	0	4	1	1	1	2	3	0.0417
0	1	1	1	0.5	1	2	1	1	0.0056
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	1	4	0	1	4	6	1	2	0.8333
1	1	4	1	1	4	6	1	3	1
1	1	4	2	1	4	6	2	1	0.6667
1	1	4	3	1	4	6	2	2	1
1	1	4	4	1	4	6	2	3	1

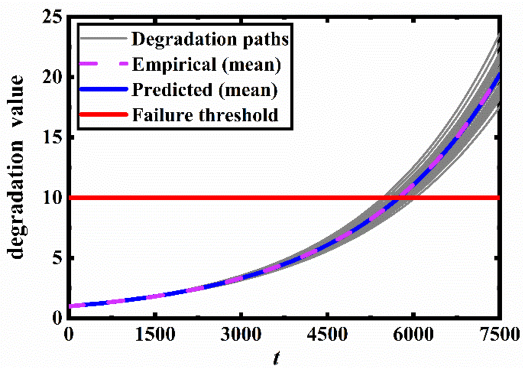
Note: The full survival signatures of these two systems are available from the authors.



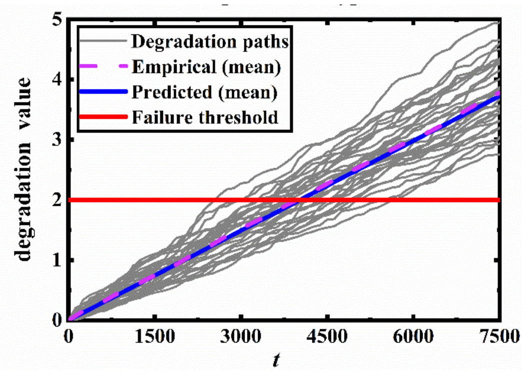
(a) Components of type one



(b) Components of type two



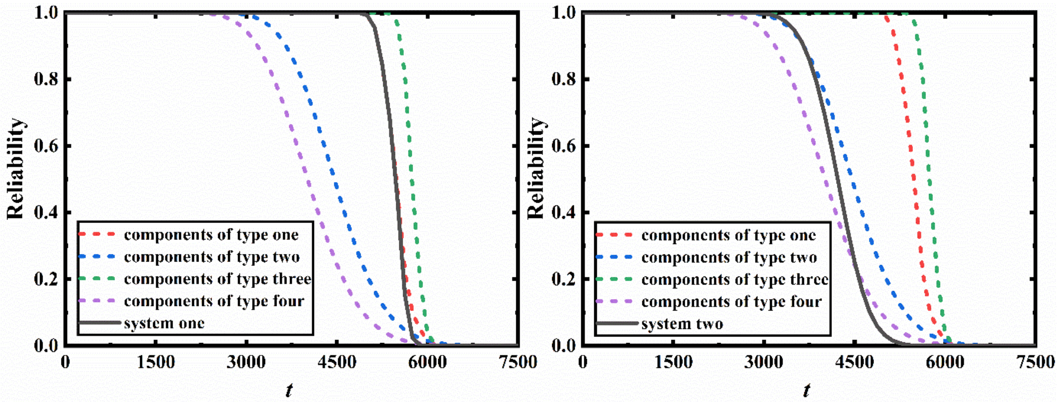
(c) Components of type three



(d) Components of type four

Fig. 13 The prediction of degradation of the four types of components

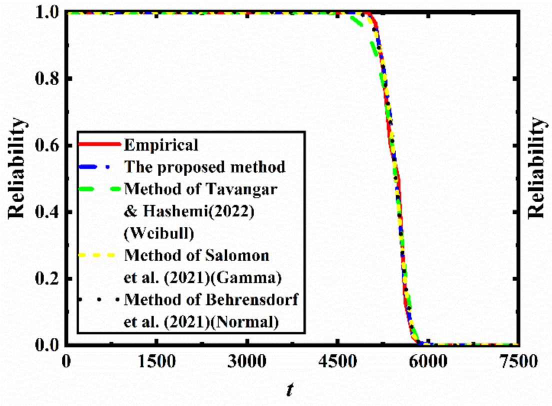
As shown in **Figs. 13**, the degradation paths of the four types of components are predicted by substituting the estimated parameters in **Table 6** into Eqs. (12-14) and Eqs. (25-26). After obtaining the values of the parameters in **Table 6** and the survival signature in **Table 8**, the reliabilities of the four types of components and of the two systems can be calculated by Eqs. (10), (23), and (31), the results are shown in **Fig. 14**. Compared with existing reliability estimation methods based on the survival signature and distributions, in this paper the system reliability analysis method is improved by modelling the degradation data with stochastic processes. In this way, not only can the reliability of the systems and components be analysed, but also the degradation levels of each type of component can be predicted, which may enable engineers to check the health conditions of systems at suitable times. In particular, for some complex systems with high-reliability requirements, time-to-failure data are not easy to collect, and failures may cause catastrophic disasters, such as the nuclear systems and building systems. In these cases, it is better to analyse the reliability and degradation level by the proposed method based on the degradation data instead of only analysing the reliability based on the failure data.



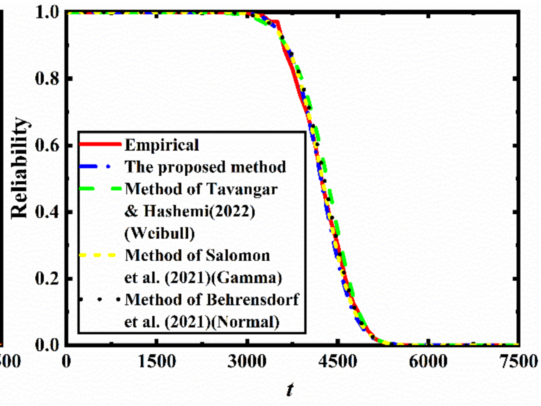
(a) system one **(b)** system two

Fig. 14 The reliability of the components and two systems

The solid red lines shown in **Fig. 15** represent the empirical reliabilities of the two systems, which are calculated by the ratio of the number of functioning components to the total number of components. The blue lines are the system reliabilities calculated by the proposed method based on the survival signature and stochastic processes. The green, yellow, and black lines represent the system reliabilities obtained by modelling the time-to-failure data with the existing methods, based on three commonly used distributions. It can be seen from **Fig. 15** that all the system reliability curves, calculated based on regardless of the stochastic processes or the distributions, fit well with the empirical system reliability curves. The MAEs between the empirical and predicted system reliabilities are shown in **Table 9**. As shown in **Fig. 15** and **Table 9**, for system one in **Fig. 10(a)**, the MAE of the predicted system reliability based on the proposed stochastic processes is the smallest, which means that the accuracy of the proposed method is the highest. For system two in **Fig. 10(b)**, the MAE of the reliability predicted by the proposed method is slightly larger than the corresponding MAE of the reliability based on the gamma distribution, but the accuracy of the system reliability predicted by the proposed method is still at a high level.



(a) reliability of system one



(b) reliability of system two

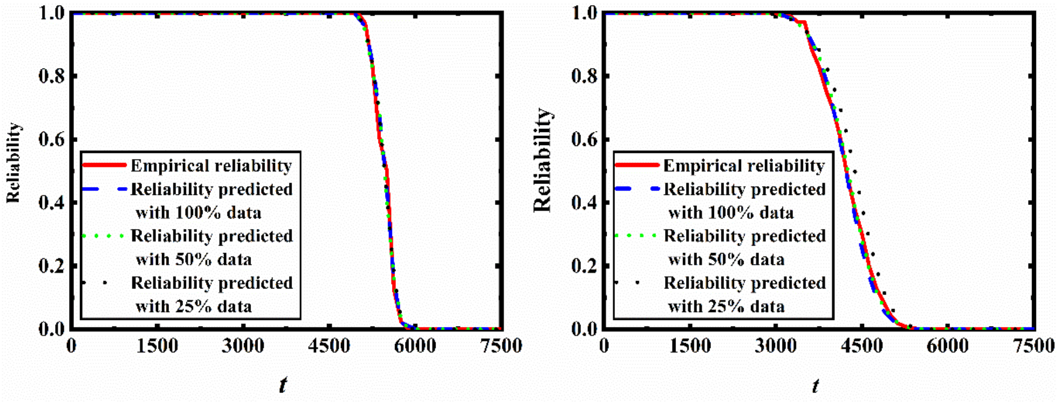
Fig. 15 The reliabilities of systems estimated by the proposed and existing methos

Table 9 The errors of the system reliabilities estimated by different methods

	The proposed method	Tavangar & Hashemi (2022)'s method (Weibull)	Salomon et al. (2021)'s method (Gamma)	Behrensdorf et al. (2021)'s method (Normal)
Reliability MAE (system one)	0.0032	0.0075	0.0041	0.0042
Reliability MAE (system two)	0.0054	0.0098	0.0045	0.0059

As shown in **Fig. 15 and Table 9**, the errors of the system reliabilities predicted based on the proposed stochastic processes are not always the smallest, but the accuracy of the proposed method is still at a high level. It is important that the system reliability predicted by the existing method based on the survival signature and distributions requires considerable amounts of failure data, which probably need to be obtained through destructive experiments. However, the system reliabilities evaluated by the survival signature and stochastic processes can be estimated based on the degradation data of the performance characteristics, such as changes in operation currents, voltages, temperatures, and vibration signals, which can be collected by sensors when the components are functioning. As shown in **Fig. 16**, the system reliability can still be predicted accurately even with only 50% degradation data available, in particular, for the first system, the system reliability can be accurately evaluated even with only 25% degradation data. Compared with the existing methods based on the survival signature and distributions, the results in **Fig. 16** show that the experiment time and expense can be greatly reduced by 50%. For less complicated systems, such as the first system in **Fig. 10(a)**, the experiment time can be reduced to

25%.



(a) reliability of system one **(b)** reliability of system two

Fig. 16 The comparison of the system reliability estimated with different amount of degradation data

5. Conclusion

System reliability research based on stochastic processes has been widely conducted, but existing methods are proposed for systems with simple structures. Therefore, a new generalised reliability model for complex systems based on stochastic processes and survival signature is proposed. The systems are not limited to simple systems, but complex systems with complicated bridge and network structures can be as easily studied, as long as the survival signature is available. In addition, in contrast to the existing system reliability analysis based on the survival signature and time-to-failure data, the proposed method based on the survival signature and degradation data can not only estimate the survival probability of the systems but also estimate the degradation level of the constituent components. The proposed method can help monitor the health conditions of systems and reduce the likelihood of disaster accidents, especially for systems whose component failures can cause catastrophic events, such as nuclear systems.

The proposed system reliability method can be widely used for complex systems with various structures, but there are still some limitations. It is applicable for components whose degradation process can be described by one key performance characteristic. However, in many practical cases, more than one performance characteristic needs to be considered when estimating reliability, such as the operational currents, temperatures, and vibration magnitudes. In addition, the measuring errors are neglected, and the systems are assumed to be unrepairable. Hence, multiple degradation processes should be taken into account for future research to improve the system reliability model based on survival signature and stochastic processes. To solve real engineering problems better, models that consider more relevant details, such as the random noise of the obtained degradation data and the repair of the components, are worthy of further exploration.

Acknowledgements

The work was supported by National Natural Science Foundation of China [grant number 51975110]; Liaoning Revitalization Talents program [grant number XLYC1907171]; and Fundamental Research Funds for the Central Universities [grant number N2003005]; China Scholarship Council [grant number 202006080071]. This research was performed when Miaoxin Chang was a visiting Ph.D. student at the Department of Mathematical Sciences, Durham University. We thank Tsai, Tseng & Balakrishnan for making the code they used in Tsai et al. (2012) available. We thank the editors and reviewers for their efforts on the review of this work.

Declaration of interest

The work described has not been published previously. The publication is approved by all authors. If accepted, it will not be published elsewhere in the same form, in English or in any other language, including electronically without the written consent of the copyright holder.

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Appendix A

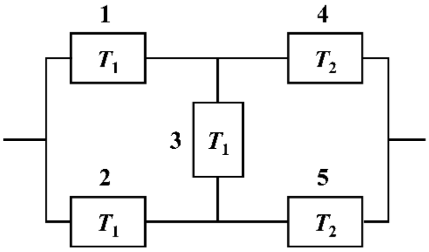

Fig. 17 A bridge system with two types of components

A bridge system with two types of components is shown in **Fig. 17**, and the survival signature of this system can be obtained by Eqs. (1-2). For the system in **Fig. 17**, there are 4×3=12 kinds of survival signatures, which are shown in **Table 10**. To better illustrate Eqs. (1-2), an example for the calculation of the vector $\boldsymbol{Y}$ and the structure function $\phi(\boldsymbol{Y})$ is provided, and the results are shown in **Table 11**.

Table 10 The survival signature of the bridge system

l_1	l_2	$\Phi_S(l_1, l_2)$	l_1	l_2	$\Phi_S(l_1, l_2)$
0	0	0	2	0	0
0	1	0	2	1	1
0	2	0	2	2	1
1	0	0	3	0	0
1	1	1/3	3	1	1
1	2	2/3	3	2	1

Table 11 The vector $\boldsymbol{Y}$ and $\phi(\boldsymbol{Y})$ when $l_1 = l_2 = 1$

<table>
<tr><th>l_1</th><th>l_2</th><th>$\boldsymbol{Y}$</th><th>$\phi(\boldsymbol{Y})$</th><th>$\Phi_S(l_1, l_2)$</th></tr>
<tr><td rowspan="6">1</td><td rowspan="6">1</td><td>(0,0,1,0,1)</td><td>0</td><td rowspan="6">1/3</td></tr>
<tr><td>(0,1,0,0,1)</td><td>1</td></tr>
<tr><td>(1,0,0,0,1)</td><td>0</td></tr>
<tr><td>(0,0,1,1,0)</td><td>0</td></tr>
<tr><td>(0,1,0,1,0)</td><td>0</td></tr>
<tr><td>(1,0,0,1,0)</td><td>1</td></tr>
</table>

Let $R_1(t)$ and $R_2(t)$ represent the reliability of the two types of components. Substituting the survival signatures in **Table 11** into Eq. (5), the system reliability can be expressed as:

$$R_s(t) = \sum_{l_1=0}^{3}\sum_{l_2=0}^{2}\left[\Phi_S(l_1,l_2)\binom{3}{l_1}R_1(t)^{l_1}\left[1-R_1(t)\right]^{3-l_1}\binom{2}{l_2}R_2(t)^{l_2}\left[1-R_2(t)\right]^{2-l_2}\right] \tag{34}$$

Appendix B

The parameter estimation procedure based on the Wiener process is as follows:

1) E-step: calculate the Q-function of by Eq. (11), which can be described as:

$$\begin{aligned} Q\left(\boldsymbol{\Theta}_{k\_\mathrm{W}} \middle| \boldsymbol{\Theta}_{k\_\mathrm{W}}^{(q)}\right) = & E\left[\ln\left(f_{\Delta\boldsymbol{X}_k|\boldsymbol{\Theta}'_k}\left(\Delta\boldsymbol{X}_k \middle| \boldsymbol{\Theta}'_k, \boldsymbol{\Theta}_{k\_\mathrm{W}}\right)\right) \middle| \Delta\boldsymbol{X}_k, \boldsymbol{\Theta}_{k\_\mathrm{W}}^{(q)}\right] \\ & + E\left[\ln\left(f_{\boldsymbol{\Theta}'_k}\left(\boldsymbol{\Theta}'_k \middle| \boldsymbol{\Theta}_{k\_\mathrm{W}}\right)\right) \middle| \Delta\boldsymbol{X}_k, \boldsymbol{\Theta}_{k\_\mathrm{W}}^{(q)}\right] \end{aligned} \tag{35}$$

where $\boldsymbol{\Theta}'_k = [\boldsymbol{\Theta}'_{1_k}, \boldsymbol{\Theta}'_{2_k}, \cdots, \boldsymbol{\Theta}'_{m_k}]$, $\boldsymbol{\Theta}'_{i_k} = [\theta_{i_k}, \eta_{i_k}]$, $i_k = 1_k, 2_k, \ldots, m_k$.

Substituting Eqs. (17-18) into Eq. (35), then the Q-function can be derived as:

$$\begin{aligned}
Q\left(\boldsymbol{\Theta}_{k_\mathrm{W}} \middle| \boldsymbol{\Theta}_{k_\mathrm{W}}^{(q)}\right) = &-\frac{m_k n_k}{2}\left(\ln 2\pi + \ln \sigma_k^2 + \ln \Delta t\right) - \frac{1}{2\sigma_k^2 \Delta t}\left[\sum_{i_k=1}^{m_k}\sum_{j_k=1}^{n_k} \Delta X_{i_k j_k}^2\right. \\
&-2\sum_{i_k=1}^{m_k}\left(\Delta X_{i_k 1_k} E\left(\theta_{i_k} \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k_\mathrm{W}}^{(q)}\right)\right) - 2\Delta t \sum_{i_k=1}^{m_k}\left(E\left(\eta_{i_k} \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k_\mathrm{W}}^{(q)}\right)\sum_{j_k=1}^{n_k}\Delta X_{i_k j_k}\right) \\
&+\sum_{i_k=1}^{m_k} E\left(\theta_{i_k}^2 \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k_\mathrm{W}}^{(q)}\right) + 2\Delta t \sum_{i_k=1}^{m_k} E\left(\theta_{i_k}\eta_{i_k} \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k_\mathrm{W}}^{(q)}\right) \\
&\left.+n_k \Delta t^2 \sum_{i_k=1}^{m_k} E\left(\eta_{i_k}^2 \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k_\mathrm{W}}^{(q)}\right)\right] - \frac{m_k}{2}\left(2\ln 2\pi + \ln \sigma_{\theta k}^2 + \ln \sigma_{\eta k}^2\right) \\
&-\frac{1}{2\sigma_{\theta k}^2}\sum_{i_k=1}^{m_k}\left(E\left(\theta_{i_k}^2 \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k_\mathrm{W}}^{(q)}\right) - 2\mu_{\theta k} E\left(\theta_{i_k} \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k_\mathrm{W}}^{(q)}\right) + \mu_{\theta k}^2\right) \\
&-\frac{1}{2\sigma_{\eta k}^2}\sum_{i_k=1}^{m_k}\left(E\left(\eta_{i_k}^2 \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k_\mathrm{W}}^{(q)}\right) - 2\mu_{\eta k} E\left(\eta_{i_k} \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k_\mathrm{W}}^{(q)}\right) + \mu_{\eta k}^2\right)
\end{aligned} \tag{36}$$

where the conditional expectations of the parameters, $\theta_{i_k}, \eta_{i_k}, \theta_{i_k}\eta_{i_k}, \theta_{i_k}^2, \eta_{i_k}^2$, can be obtained according to Si et al. (2013).

$$\begin{cases}
E\left(\theta_{i_k} \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k_\mathrm{W}}^{(q)}\right) = \mu'_{\theta i_k} \\
E\left(\eta_{i_k} \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k_\mathrm{W}}^{(q)}\right) = \mu'_{\eta i_k} \\
E\left(\theta_{i_k}^2 \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k_\mathrm{W}}^{(q)}\right) = \mu'^2_{\theta i_k} + \sigma'^2_{\theta i_k} \\
E\left(\eta_{i_k}^2 \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k_\mathrm{W}}^{(q)}\right) = \mu'^2_{\eta i_k} + \sigma'^2_{\eta i_k} \\
E\left(\theta_{i_k}\eta_{i_k} \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k_\mathrm{W}}^{(q)}\right) = \rho_{i_k}\sigma'_{\theta i_k}\sigma'_{\eta i_k} + \mu'_{\theta i_k}\mu'_{\eta i_k}
\end{cases} \tag{37}$$

$$\begin{cases}
\mu'_{\theta i_k} = \dfrac{\left(\sigma_{\theta k}^{(q)2}\Delta X_{i_k 1_k} + \mu_{\theta k}^{(q)}\sigma_k^{(q)2}\Delta t\right)\left(\sigma_{\eta k}^{(q)2} t_{n_k} + \sigma_k^{(q)2}\Delta t\right) - \sigma_{\theta k}^{(q)2}\Delta t\left(\sigma_{\eta k}^{(q)2} X_{i_k}\left(t_{n_k}\right) + \mu_{\eta k}^{(q)}\sigma_k^{(q)2}\right)}{\left(\sigma_{\theta k}^{(q)2} + \sigma_k^{(q)2}\Delta t\right)\left(\sigma_{\eta k}^{(q)2} t_{n_k} + \sigma_k^{(q)2}\right) - \sigma_{\theta k}^{(q)2}\sigma_{\eta k}^{(q)2}\Delta t} \\
\sigma'^2_{\theta i_k} = \dfrac{\sigma_{\theta k}^{(q)2}\sigma_k^{(q)2}\Delta t\left(\sigma_k^{(q)2} + \sigma_{\eta k}^{(q)2} t_{n_k}\right)}{\left(\sigma_{\theta k}^{(q)2} + \sigma_k^{(q)2}\Delta t\right)\left(\sigma_{\eta k}^{(q)2} t_{n_k} + \sigma_k^{(q)2}\right) - \sigma_{\theta k}^{(q)2}\sigma_{\eta k}^{(q)2}\Delta t} \\
\mu'_{\eta i_k} = \dfrac{\left(X_{i_k}\left(t_{n_k}\right)\sigma_{\eta k}^{(q)2} + \mu_{\eta k}^{(q)}\sigma_k^{(q)2}\right)\left(\sigma_{\theta k}^{(q)2} + \sigma_k^{(q)2}\Delta t\right) - \sigma_{\eta k}^{(q)2}\left(\sigma_{\theta k}^{(q)2}\Delta X_{i_k 1_k} + \mu_{\theta k}^{(q)}\sigma_k^{(q)2}\Delta t\right)}{\left(\sigma_{\theta k}^{(q)2} + \sigma_k^{(q)2}\Delta t\right)\left(\sigma_{\eta k}^{(q)2} t_{n_k} + \sigma_k^{(q)2}\right) - \sigma_{\theta k}^{(q)2}\sigma_{\eta k}^{(q)2}\Delta t} \\
\sigma'^2_{\eta i_k} = \dfrac{\sigma_{\eta k}^{(q)2}\sigma_k^{(q)2}\left(\sigma_{\theta k}^{(q)2} + \sigma_k^{(q)2}\Delta t\right)}{\left(\sigma_{\theta k}^{(q)2} + \sigma_k^{(q)2}\Delta t\right)\left(\sigma_{\eta k}^{(q)2} t_{n_k} + \sigma_k^{(q)2}\right) - \sigma_{\theta k}^{(q)2}\sigma_{\eta k}^{(q)2}\Delta t} \\
\rho_{i_k} = \dfrac{-\sigma_{\theta k}^{(q)}\sigma_{\eta k}^{(q)}\sqrt{\Delta t}}{\sqrt{\left(\sigma_{\theta k}^{(q)2} + \sigma_k^{(q)2}\Delta t\right)\left(\sigma_{\eta k}^{(q)2} t_{n_k} + \sigma_k^{(q)2}\right)}}
\end{cases} \tag{38}$$

2) **M-Step:** update $\boldsymbol{\Theta}_{k\_\mathrm{W}}^{(q+1)}$ as $\underset{\boldsymbol{\Theta}_{k\_\mathrm{W}}}{\arg\max}\, \partial Q(\boldsymbol{\Theta}_{k\_\mathrm{W}} \mid \boldsymbol{\Theta}_{k\_\mathrm{W}}^{(q)})$.

Let $\partial Q(\boldsymbol{\Theta}_{k\_\mathrm{W}} \mid \boldsymbol{\Theta}_{k\_\mathrm{W}}^{(q)}) / \partial \boldsymbol{\Theta}_{k\_\mathrm{W}} = 0$, and the estimation of deterministic parameters in the q+1 step can be expressed as Eq. (24).

Appendix C

The parameter estimation procedure based on the Gamma process is as follows:

1) **E-step:** the Q-function calculated by Eq. (10) can be described as:

$$\begin{aligned} Q\left(\boldsymbol{\Theta}_{k\_\mathrm{G}} \middle| \boldsymbol{\Theta}_{k\_\mathrm{G}}^{(q)}\right) &= E\left[\ln\left(f_{\boldsymbol{\Delta X}_k|\boldsymbol{\Theta}'_k}\left(\boldsymbol{\Delta X}_k \middle| \boldsymbol{\Theta}'_k, \boldsymbol{\Theta}_{k\_\mathrm{G}}\right)\right) \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k\_\mathrm{G}}^{(q)}\right] \\ &\quad + E\left[\ln\left(f_{\boldsymbol{\Theta}'_k}\left(\boldsymbol{\Theta}'_k \middle| \boldsymbol{\Theta}_{k\_\mathrm{G}}\right)\right) \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k\_\mathrm{G}}^{(q)}\right] \end{aligned} \tag{39}$$

where $\boldsymbol{\Theta}'_k = [\boldsymbol{\Theta}'_{1_k}, \boldsymbol{\Theta}'_{2_k}, \cdots, \boldsymbol{\Theta}'_{m_k}] = [\beta_{1_k}, \beta_{2_k}, \cdots, \beta_{m_k}]$.

Substituting Eqs. (27-28) into Eq. (39), then the Q-function can be derived as:

$$\begin{aligned} Q\left(\boldsymbol{\Theta}_{k\_\mathrm{G}} \middle| \boldsymbol{\Theta}_{k\_\mathrm{G}}^{(q)}\right) &= \left(a_k \Delta t - 1\right) \sum_{i_k=1}^{m_k} \sum_{j_k=1}^{n_k} \ln \Delta X_{i_k j_k} + \left(n_k a_k \Delta t + \delta_k - 1\right) \sum_{i_k=1}^{m_k} E\left(\ln \beta_{i_k} \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k\_\mathrm{G}}^{(q)}\right) \\ &\quad - \sum_{i_k=1}^{m_k} \sum_{j_k=1}^{n_k} \Delta X_{i_k j_k} E\left(\beta_{i_k} \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k\_\mathrm{G}}^{(q)}\right) - m_k n_k \ln \Gamma\left(a_k \Delta t\right) + m_k \delta_k \ln \lambda_k \\ &\quad - \lambda_k \sum_{i_k=1}^{m_k} E\left(\beta_{i_k} \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k\_\mathrm{G}}^{(q)}\right) - m_k \ln \Gamma\left(\delta_k\right) \end{aligned} \tag{40}$$

The conditional expectations of the parameters, β_{i_k} and $\ln \beta_{i_k}$, can be obtained according to Ye et al. (2014) and Tsai et al. (2012):

$$E\left[\beta_{i_k} \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k\_\mathrm{G}}^{(q)}\right] = \frac{n_k a_k^{(q)} \Delta t + \delta_k^{(q)}}{\sum_{j_k=1}^{n_k} \Delta X_{k_k j_k} + \lambda_k^{(q)}} \tag{41}$$

$$E\left[\ln \beta_{i_k} \middle| \boldsymbol{\Delta X}_k, \boldsymbol{\Theta}_{k\_\mathrm{G}}^{(q)}\right] = \Psi\left(n_k a_k^{(q)} \Delta t + \delta_k^{(q)}\right) - \ln\left(\sum_{j_k=1}^{n_k} \Delta X_{i_k j_k} + \lambda_k^{(q)}\right) \tag{42}$$

where, $\Psi(x)$ is the digamma function, $\Psi(x) = \Gamma'(x) / \Gamma(x)$.

2) **M-Step:** update $\boldsymbol{\Theta}_{k\_\mathrm{G}}^{(q+1)}$ as $\underset{\boldsymbol{\Theta}_{k\_\mathrm{G}}}{\arg\max}\, \partial Q(\boldsymbol{\Theta}_{k\_\mathrm{G}} \mid \boldsymbol{\Theta}_{k\_\mathrm{G}}^{(q)})$.

Let $\partial Q(\boldsymbol{\Theta}_{k\_\mathrm{G}} \mid \boldsymbol{\Theta}_{k\_G}^{(q)}) / \partial \boldsymbol{\Theta}_{k\_\mathrm{G}} = 0$, and the estimation of deterministic parameters of the Gamma process in the q+1 step can be expressed as Eq. (32).

Appendix D

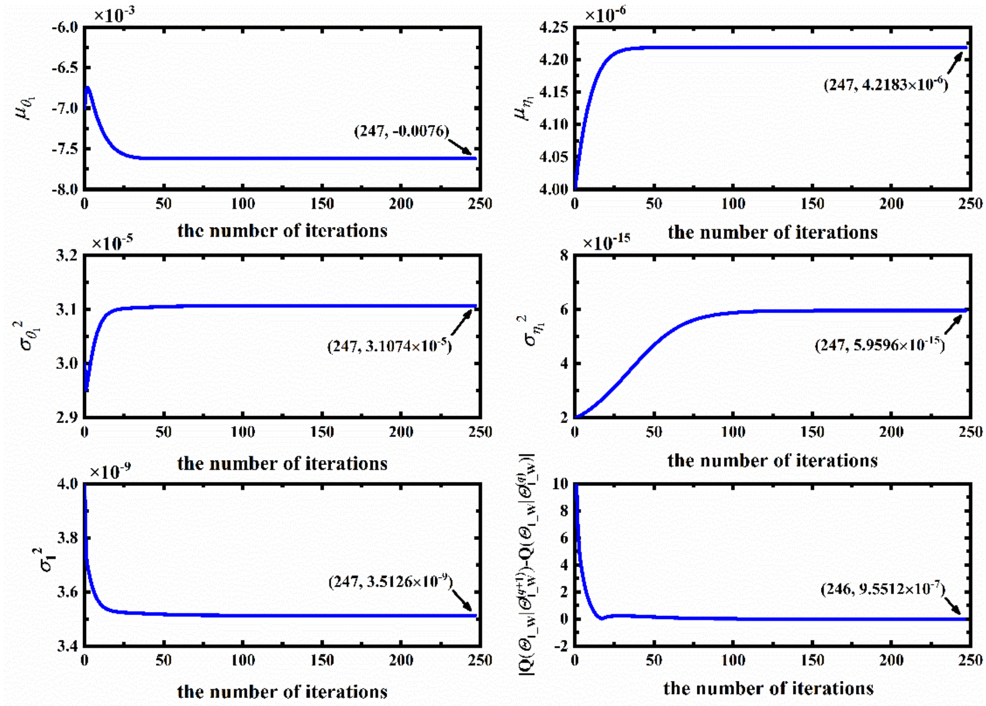

Fig. 18 The parameter estimation results of the Wiener process based on the fatigue crack data

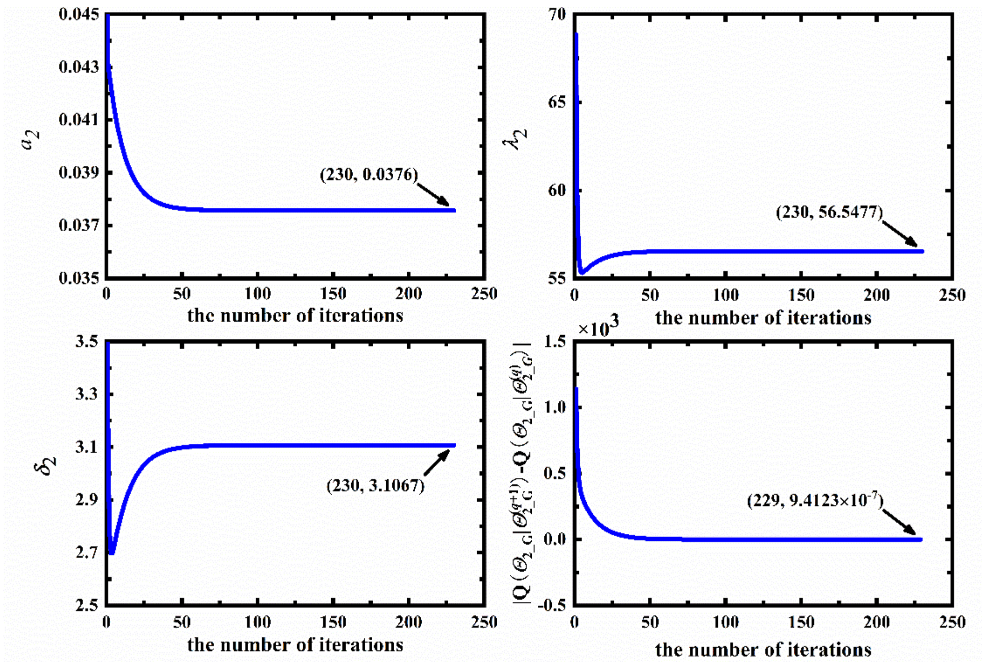

Fig. 19 The parameter estimation results of the Gamma process based on the operating currents data