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RESEARCH ARTICLE

New weighted rank correlation coefficients sensitive to agreement on top and bottom rankings

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Three new weighted rank correlation coefficients are proposed which are sensitive to both agreement on top and bottom rankings. The first one is based on the weighted rank correlation coefficient proposed by Maturi and Abdelfattah [13], the second and the third are based on the order statistics and the quantiles of the Laplace distribution, respectively. The limiting distributions of the new correlation coefficients under the null hypothesis of no association between the rankings are presented, and a summary of the exact and approximate quantiles for these coefficients is provided. A simulation study is performed to compare the performance of Kendall's tau, Spearman's rho, and the new weighted rank correlation coefficients in detecting the agreement on the top and the bottom rankings simultaneously. Finally, examples are given for illustration purposes, including a real data set from financial market indices.

Keywords: Weighted correlation; Measures of agreement; Measures of association; Rank correlation; Laplace distribution, Order statistics.

1. Introduction

There are many situations where n objects are ranked by two independent sources or observers and the interest is in measuring the agreement between these sets of rankings. Measures of agreement or association are of much interest in practice, for example to evaluate the agreement between several experts, methods or models.

In many cases the interest is particularly focused on agreement on the top and bottom rankings (lower and upper rankings), as e.g. in the study of the agreement between preference rankings, the agreement between football ranking league tables, evaluate the agreement between two methods of assessing platelet aggregation [21], and the agreement between financial markets [16]. There maybe situations where judges believed to give more attention to the allocation of the top and bottom rankings, than the middle ranks (they may even randomly allocate ranks to the middle subjects). In this case, an analyst may wish to focus on the agreement between judges with respect to the top and bottom rankings [4].

Statistics such as Spearman's rho [18] or Kendall's tau [10] correlation coefficients are not appropriate for such a scenario since they assign equal weights to all rankings. Several correlation coefficients have been proposed in the literature which are more sensitive to the agreement on the top rankings, such as the top-down correlation coefficient by Iman and Conover [9] which is based on Savage scores, the weighted Kendall's tau by Shieh [17], the Blest's correlation coefficient [2] and its

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symmetric version [5]. Maturi and Abdelfattah [13] presented a weighted rank correlation coefficient with the flexibility of choosing weights to reflect the emphasis of agreement on the top rankings.

This paper presents three new weighted rank correlation coefficients which are sensitive to agreement on top and bottom rankings simultaneously. The first one is based on the weights that are proposed by Maturi and Abdelfattah [13] which is more sensitive to the agreement on the top and bottom rankings simultaneously. The second and the third are based on the order statistics and the quantiles of the Laplace distribution, respectively.

The rest of the paper is organized as follows. The three new weighted rank correlation coefficients are introduced in Section 2. The limiting distributions of these correlation coefficients under the null hypothesis of no association among the rankings are presented in Section 3. A summary of the exact and approximate quantiles for these correlation coefficients is also provided in this section. A simulation study has been performed in order to investigate the performance of the new correlation coefficients compared to Spearman's rho and Kendall's tau correlation coefficients, the results are presented in Section 4. In order to illustrate the important features of the new correlation coefficients, examples are given in Section 5, including a real data set from financial market indices. Some concluding remarks are given in Section 6.

2. New weighted rank correlation coefficients

In this section we introduce three weighted rank correlation coefficients to assess the agreement on the top and bottom rankings simultaneously. The first weighted rank correlation coefficient is based on the weight scores introduced by Maturi and Abdelfattah [13]. The weighted correlation coefficient proposed by Maturi and Abdelfattah [13] is sensitive to the agreement on the top (or bottom) rankings but not on the top and bottom simultaneously, also their rank correlation is not symmetric. Coolen-Maturi [3] extended the rank correlation in [13] for more than two sets of rankings but again the focus was only on the agreement on the top or bottom rankings. The new weighted rank correlation presented in Section 2.1, uses the weight scores in [13], is symmetric and can be used to evaluate the agreement on the top and the bottom rankings at the same time. In addition it enjoys the same flexibility as the one in [13] in terms of choice of weight scores which take the value between 0 and 1, exclusive.

Two further weighted rank correlation coefficients are presented in Section 2.2. These are based on the Laplace distribution which is also known as the Double Exponential distribution, because it can be thought of as the distribution of the difference of two independent identically distributed exponential random variables [11]. The first coefficient is based on the order statistics of the Laplace distribution while the second is based on the quantiles of the Laplace distribution. The performance of these three coefficient will be considered in Section 4.

Throughout this paper, we assume that there are no ties among rankings, however if ties occur we suggest to use the randomization tool [6] to deal with ties as the randomization methods do not affect the null distribution of the rank correlation coefficients [6, 14].

2.1 *A new weighted correlation coefficient based on Maturi and Abdelfattah [13] scores*

Maturi and Abdelfattah [13] introduced a weighted rank correlation coefficient, r_w, to test the null hypothesis that two set of rankings are independent. This weighted rank correlation r_w is sensitive to agreement on the top ranks, it is based on the weighted scores $(w^{R_{1i}}, w^{R_{2i}})$ where (R_{1i}, R_{2i}) are the paired rankings of object $i = 1, 2, \ldots, n$, with weight $w \in (0, 1)$.

The weighted rank coefficient r_w is given by [13]

$$r_w = \left(n \sum_{i=1}^{n} w^{R_{1i}+R_{2i}} - a_1 \right) / (na_2 - a_1) \tag{1}$$

where $a_1 = w^2(1 - w^n)^2/(1 - w)^2$ and $a_2 = w^2(1 - w^{2n})/(1 - w^2)$. The statistic r_w has a maximum value of 1, yet its minimum possible value is not -1. In fact, the minimum value of R_w is -1 only for $n = 2$ and increases away from -1 towards approximately from -2×10^{-6} to -3×10^{-4}, depending on the value of w. This is very similar behaviour to the top-down correlation coefficient introduced by Iman and Conover [9]. Maturi and Abdelfattah [13] showed that r_w is a locally most powerful rank test. For $n \to \infty$ and under the null hypothesis of independence, they showed that the statistic $r_w\sqrt{n-1}$ has asymptotically a standard normal distribution.

We will use the weighted scores proposed by Maturi and Abdelfattah [13] to derive the new correlation coefficient, and by using the notation introduced above, the weighted scores are $w^{R_{ji}}$ where R_{ji} is the rank given by the jth observer to the ith object ($j = 1, 2$ and $i = 1, 2, \ldots, n$) and $0 < w < 1$. The choice of w reflects the desire to emphasize the lower and upper rankings. For ease of presentation and without loss of generality, let q_i be the rank given by the second observer corresponding to the rank i given by the first observer. That is we have paired rankings (i, q_i), $i = 1, 2, \ldots, n$, of n objects. The new weighted scores that give more weights to the lower and upper rankings are defined as

$$S_{w_i} = \begin{cases} -w^i & \text{if} \quad i < \frac{n+1}{2} \\ 0 & \text{if} \quad i = \frac{n+1}{2} \\ w^{n+1-i} & \text{if} \quad i > \frac{n+1}{2} \end{cases} \tag{2}$$

For example, from Table 1, the scores S_{w_i} for the rankings 1,2,3,4,5 ($n = 5$) and $w = 0.3$ are $-0.30, -0.09, 0, 0.09, 0.30$. Notice that the scores S_{w_i} for $n = 4$ and $w = 0.3$ are $-0.30, -0.09, 0.09, 0.30$.

The new weighted rank coefficient R_w is defined by computing the well-known Pearson correlation coefficient on the weighted scores in (2) as

$$R_w = \frac{\sum_{i=1}^{n} S_{w_i} S_{w_{q_i}}}{\sqrt{\sum_{i=1}^{n} S_{w_i}^2} \sqrt{\sum_{i=1}^{n} S_{w_{q_i}}^2}} \tag{3}$$

which is equal to

$$R_w = \frac{1}{A} \sum_{i=1}^{n} S_{w_i} S_{w_{q_i}} \tag{4}$$

where $A = \frac{2w^2(1-w^{2m})}{1-w^2}$ and $m = \lceil \frac{n-1}{2} \rceil$, as shown in the proof of Theorem 3.1.

2.2 *Two correlation coefficients based on the Laplace distribution*

Iman and Conover [9] proposed the top-down correlation coefficient, by computing the well-known Pearson correlation coefficient on Savage scores ($S_i = \sum_{l=i}^{n} 1/l$), which is more sensitive to the agreement on the top rankings. The statistic that uses Savage scores is shown to be asymptotically normal under the null hypothesis of independence and to be a locally most powerful test for some alternatives, for more details see [8, 9]. However, the top-down correlation coefficient is not symmetric, one can make it symmetric by taking $-1+S_i = -1+\sum_{l=i}^{n} 1/l$, but this is not fully symmetric.

The savage score S_{n-i+1} can also be defined as the expected value of the ith order statistic in a random sample of size n from the exponential distribution [8, p. 78]. This leads to the idea of a new correlation coefficient which is based on the following scores

$$S_{O_i} = \frac{1}{2^n}\left\{\sum_{r=1}^{i}\binom{n}{r-1}E(i-r+1, n-r+1) - \sum_{r=i}^{n}\binom{n}{r}E(r-i+1, r)\right\}$$

where $E(a, b) = \sum_{l=1}^{a} \frac{1}{b-l+1}$ for any integers $a < b$. The S_{O_i} score is the expected value of the ith order statistic in a random sample of size n from the Laplace distribution (also known as Double Exponential distribution) [1, 7]. These scores S_{O_i} are symmetric around the middle ranks. A new rank correlation coefficient R_O is obtained by computing the Pearson correlation coefficient on scores S_{O_i},

$$R_O = \frac{\sum_{i=1}^{n} S_{O_i} S_{O_{q_i}}}{\sqrt{\sum_{i=1}^{n} S_{O_i}^2}\sqrt{\sum_{i=1}^{n} S_{O_{q_i}}^2}} \tag{5}$$

This rank correlation coefficient takes the values between -1 and 1, inclusive, with -1 (+1) for perfect negative (positive) correlation. We should emphasize that the rank correlation coefficient R_O gives more weight to the lower and upper rankings simultaneously. For example, from Table 1, the scores S_{O_i} for the rankings 1,2,3,4,5 ($n = 5$) are $-1.5885, -0.5729, 0, 0.5729, 1.5885$.

A second new correlation coefficient can also be based on the Laplace distribution, namely on the quantiles of Laplace distribution. We define the score for the rank i as

$$S_{L_i} = F_L^{-1}\left(\frac{i}{n+1}\right)$$

where F_L^{-1} is the inverse cumulative distribution function of Laplace distribution. These scores can be easily obtained using any statistical software, e.g. the package '`rmutil`' [1] in R. These scores S_{L_i} are symmetric around the middle ranks, for example, from Table 1, the scores S_{L_i} for the rankings 1,2,3,4,5 ($n = 5$) are $-1.0986, -0.4055, 0, 0.4055, 1.0986$.

The rank correlation coefficient R_L is defined by computing the Pearson corre-

[1]http://www.commanster.eu/rcode.html

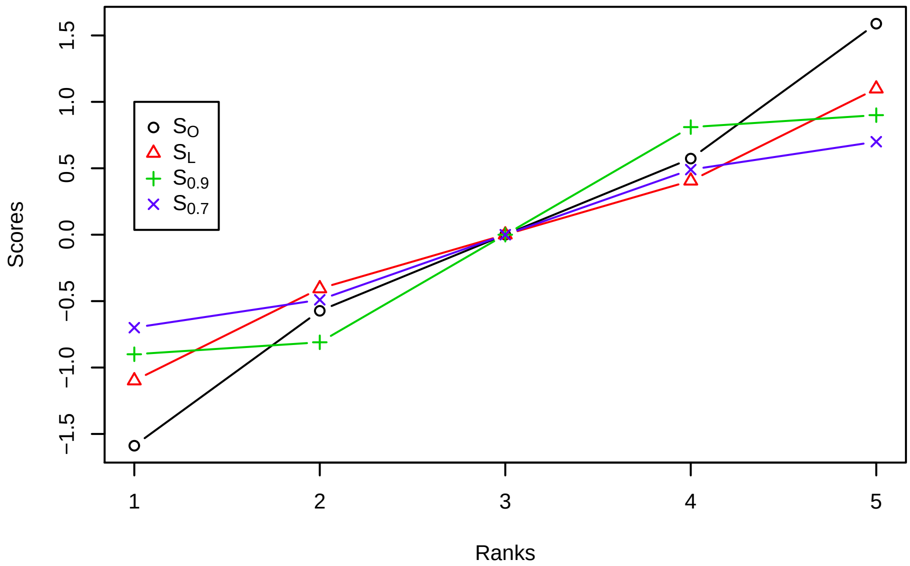

Figure 1. Scores for $n = 5$

lation coefficient on scores S_{L_i},

$$R_L = \frac{\sum_{i=1}^{n} S_{L_i} S_{L_{q_i}}}{\sqrt{\sum_{i=1}^{n} S_{L_i}^2}\sqrt{\sum_{i=1}^{n} S_{L_{q_i}}^2}} \tag{6}$$

Like R_O, the rank correlation coefficient R_L is symmetric and gives more weight to the lower and upper rankings simultaneously. R_L is easy to calculate using statistical packages and, as shown in Section 4, its power is close to the power of the correlation coefficient R_O.

To summarize, all three proposed weighted rank correlation coefficients are symmetric, taking the value -1 for prefect negative agreement and +1 for prefect positive agreement. All give more weights to the lower and upper rankings simultaneously. To illustrate the different weights that are given by the three new rank correlation coefficients, consider the weight of the ranks 1,2,3,4,5 as given in Table 1 and Figure 1. Obviously, the scores S_O and S_L assign more weights to the lower and upper rankings compared to $R_{0.7}$ and $R_{0.9}$.

3. The exact and limiting distributions

In order to use the proposed weighted rank correlation coefficients to test the null hypothesis of no agreement between the two rankings, one needs to find the distribution of these weighted rank correlation coefficients under the null hypothesis. The exact quantiles of R_w for $n = 4(1)12$, and the approximate quantiles for larger n ($n = 13(1)19$ and $n = 20(10)100$) are listed in Tables 2, 3 and 4. The exact quantiles were obtained by generating all possible permutations of $(1, 2, \ldots, n)$, then the three proposed rank correlation coefficients are calculated between $(1, 2, \ldots, n)$ and each of these (equally likely) permutations. The approximate quantiles were obtained by one million Monte Carlo simulations for each n. Similarly, the exact and approximate quantiles of R_L and R_O are reported in Tables 5 and 6, respectively.

For large values of n, the asymptotic distribution of R_w under the null hypothesis is given in the following theorem. The three weighted rank correlation coefficients have the same asymptotic distribution, that is the limiting distribution of R_w, R_L and R_O is the normal distribution with mean 0 and variance $1/(n-1)$. The proof for R_w is given below, the proof for the other two weighted rank correlation coefficients, R_L and R_O, is very similar.

THEOREM 3.1 *Under the null hypothesis of independence, $E(R_w) = 0$, $V(R_w) = 1/(n-1)$ and the asymptotic distribution of $R_w\sqrt{n-1}$ is the standard normal distribution.*

Proof
The mean and the variance of the R_w, under null hypothesis of independence, are computed as follows. Since $E(S_{w_i} S_{w_{q_i}}) = E(S_{w_i})E(S_{w_{q_i}}) = 0$ then by substituting in (4) we directly obtain that $E(R_w) = 0$. Let $m = \lceil \frac{n-1}{2} \rceil$, from (2),

$$A = \sum_{i=1}^{n} S^2_{w_i} = \sum_{i} (-w^i)^2 + \sum_{i} (w^{n+1-i})^2 = 2\sum_{i=1}^{m} w^{2i} = \frac{2w^2(1-w^{2m})}{1-w^2}$$

For the variance, from (4),

$$Var(R_w) = \frac{1}{A^2} Var(\sum_{i=1}^{n} S_{w_i} S_{w_{q_i}})$$

where

$$Var(\sum_{i=1}^{n} S_{w_i} S_{w_{q_i}}) = nVar(S_{w_i})Var(S_{w_{q_i}}) + n(n-1)Cov(S_{w_i}, S_{w_j})Cov(S_{w_{q_i}}, S_{w_{q_j}})$$

and

$$Var(S_{w_i}) = Var(S_{w_{q_i}}) = \frac{1}{n}\sum_{i=1}^{n} S^2_{w_i} = \frac{1}{n}\sum_{i=1}^{n} S^2_{w_{q_i}} = \frac{1}{n}A$$

$$Cov(S_{w_i}, S_{w_j}) = Cov(S_{w_{q_i}}, S_{w_{q_j}}) = E(S_{w_i}S_{w_j}) - E(S_{w_i})E(S_{w_j})$$
$$= \frac{1}{n(n-1)} \sum_{i \neq j} S_{w_i}S_{w_j} - 0$$
$$= \frac{1}{n(n-1)} \left(\left(\sum_{i=1}^{n} S_{w_i} \right)^2 - \sum_{i=1}^{n} S_{w_i}^2 \right) = \frac{-A}{n(n-1)}$$

then

$$Var(\sum_{i=1}^{n} S_{w_i}S_{w_{q_i}}) = n\left(\frac{A}{n}\right)^2 + n(n-1)\left(\frac{-A}{n(n-1)}\right)^2 = \frac{A^2}{n-1}$$

therefore $Var(R_w) = 1/(n-1)$.

Using $a_n(R_{ni}, f) = S_{w_i}/\sqrt{A}$ and $a_n(Q_{ni}, g) = S_{w_{q_i}}/\sqrt{A}$, we can write $R_w = \sum_{i=1}^{n} a_n(R_{ni}, f)a_n(Q_{ni}, g)$. That is, R_w is written as a linear rank statistic. Under H_0, using Theorem V.1.8 in [8], the distribution of the statistic R_w for $n \to \infty$ is asymptotically normal with mean 0 and variance $1/(n-1)$. ■

4. Simulation study

A simulation study has been carried out to compare the performance of the three new correlation coefficients, Kendall's tau and Spearman's rho and in detecting the agreement on top and bottom rankings simultaneously. To compare the power of the three new weighted rank correlation coefficients we followed Legendre [12] in generating the following simulation scenario[1].

For the first set of observations, a standard normal distribution sample is generated then this sample is sorted ascendingly. To obtain the second sample, (1) two random normal distribution samples (of size $m = [np]$ where $0 < p < 1$) with mean zero and standard deviations $\sigma = 0.25, 0.5, 1, 2, 3$ are simulated and then added to the first (sorted) set of observations, for $i = 1, \ldots, m$ and $i = n-m+1, \ldots n$, (2) a random normal distribution sample (of size $n - 2m$) with mean zero and standard deviations $\sigma = 0.25, 0.5, 1, 2, 3$ is simulated for $i = m+1, \ldots, n-m$. Three values of the proportions of ranks are considered here, namely $p = 0.1, 0.2, 0.3$, which allow us to compare the performance of the new correlations coefficients at different levels of focus on the top and bottom rankings simultaneously.

The performances (power) of the three weighted correlation coefficients are assessed, at significance level $\alpha = 0.05$, using the percentage of rejections of the null hypothesis when the null hypothesis is false. For the rank correlation coefficient R_w we are going to consider the weights $w = 0.4(0.1)0.8$. Based on 10,000 replications and $n = 10, 20, 30, 50, 100$, the simulation results are summarised in Table 7.

From Table 7 and as expected all the new weighted rank correlation coefficients perform much better compared with Spearman's rho and Kendall's tau for $p = 0.1$ than for $p = 0.3$. And of course the powers are much higher for small σ where larger values of σ correspond to lower degrees of agreement.

Table 7 also shows that the Spearman's rho performs slightly better than Kendall's tau but both are very close. The two weighted rank correlation coefficients that are based on the Laplace distribution, R_L and R_O, perform very well

[1]The R codes are available on request from the author.

compared to Spearman's rho and Kendall's tau. The performance of R_L and R_O is very close to each other, however R_O performs better than R_L for $p = 0.1$ while R_L performs slightly better than R_O for $p = 0.3$.

The performance of the weighted rank correlation R_w varies depending on the sample size n, σ and p. For small p, σ and n, the R_w (for small w) performs better than other correlation coefficients. For large n, R_w performs better than (or close to) other correlation coefficients, except for large variance σ. This is not surprising, as for R_w one has huge freedom on how to choose the weight to reflect the amount of emphasis on agreement on top and bottom rankings. Therefore one should choose the weights very carefully taking into account the research interest and sample size.

As a rule of thumb, if one wishes to find the agreement on 10% of both the top and bottom rankings without giving much attention to the ranks on the middle then one may choose small values of w, e.g. $w \leq 0.3$. And if one would like to find the agreement on 20% of both the top and bottom rankings with some attention to the values in the middle then one may want to choose medium values of w, e.g. between 0.4 and 0.6. However, if the aim is to find the agreement on the 30-35% of the top and bottom rankings with reasonable attention to the values in the middle then the suggestion is to choose large values of w, e.g. between 0.7 and 0.9. Figure 2 shows the weighted scores S_w for $n = 15, 25, 50$, which illustrates how the weighted scores change from the top/bottom rankings compared with the middle ranks.

5. Examples

In this section, two examples are presented to illustrate the new proposed weighted rank correlation coefficients, the second example considers real data on financial markets indices.

Example 5.1 Let us consider seven scenarios in which two experts are asked to rank 15 objects ($n = 15$) according to some criteria, this data set is given in Table 8. We calculate the three proposed rank correlation coefficients (using (3), (5) and (6)) along with Kendall's tau and Spearman's rho, the results are summarized in Table 9. Scenarios (A,B) and (A,C) show almost perfect positive agreement on the top and bottom rankings (4 rankings both sides) while scenario (A,E) shows almost perfect negative agreement on the top and bottom rankings (4 rankings both sides). For scenario (A,B), all the rank correlation coefficients are significant at significance level 1%, except $R_{0.9}$ which is significant at 5%, with large values for all the weighted rank correlations, except $R_{0.8}$ and $R_{0.9}$. The merit of the new proposed weighted rank correlation coefficients is more obvious in scenario (A,D) where we have perfect positive agreement on the top and bottom rankings (2 rankings both sides). For this scenario, Kendall's tau, Spearman's rho and $R_{0.9}$ are significant at significance level 5% while the remaining correlation coefficients are significant at 1%.

On the other hand, in Scenarios (A,F), (A,G) and (A,H) we have perfect agreement on the middle rankings, while there is some disagreement on the top and bottom rankings (2, 3, 4 rankings both sides, respectively). All rank correlation coefficients are significant at significance level 1%, except for R_w with small values of w which are not significant. This shows the impact of disagreement on the top and bottom rankings on R_w for small values of w. It also shows how choosing the wights for R_w could affect the inference results.

Example 5.2 In this example a set of monthly data of four market indices is used to illustrate the new rank correlation coefficients, these indices are the Standard

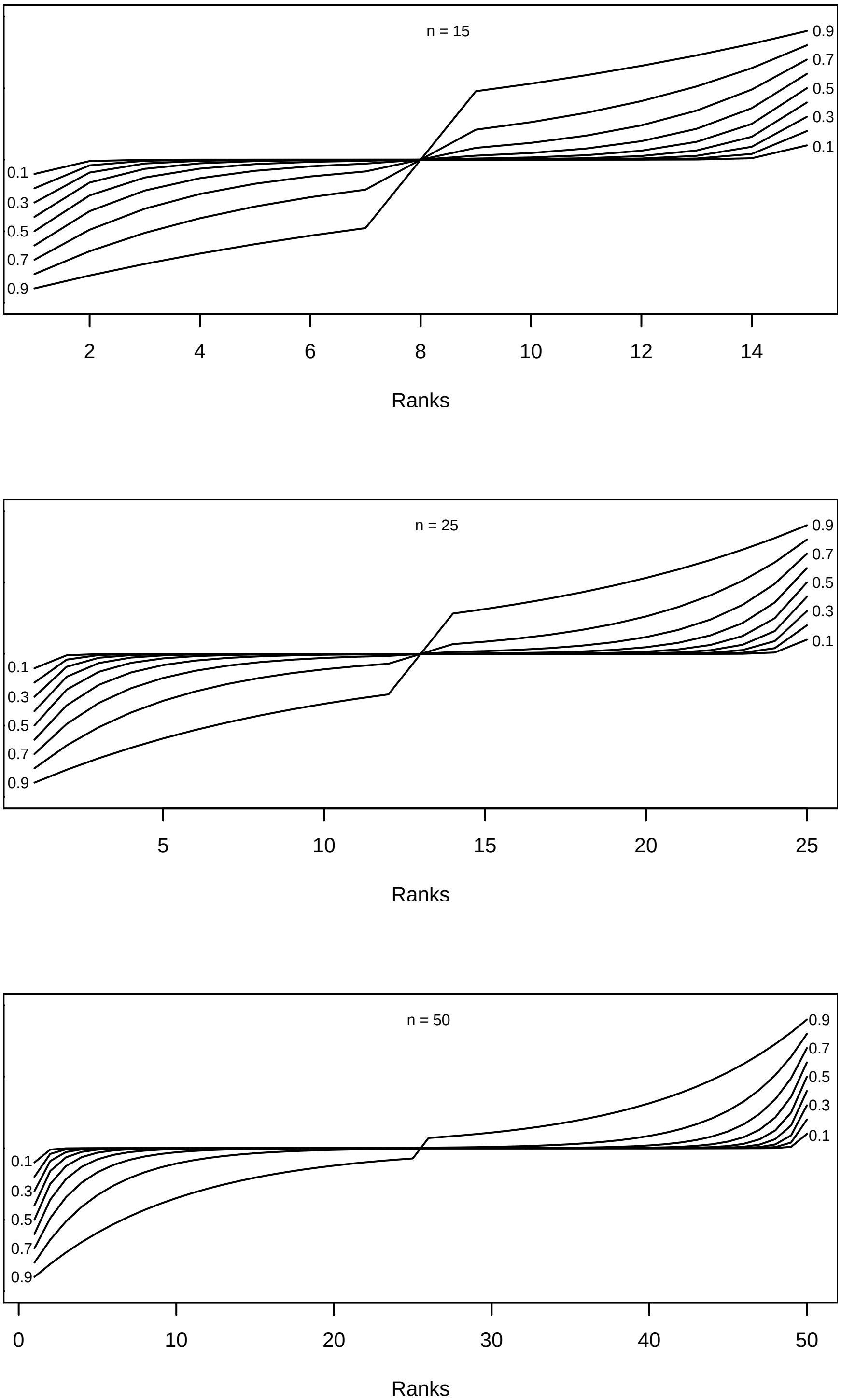

Figure 2. S_w scores for $n = 15$, $n = 25$ and $n = 50$

and Poor (S&P), the Financial Times (FT), the Nikkei (Nik), and the DAX index. Meintanis and Iliopoulos [15] used this data set to test the independence of the four indices as well as of all combinations of three or two of them. Coolen-Maturi [3] also used this data set to test the independence of the four indices with more focus on agreement on the top rankings via the weighted rank coefficient of concordance,

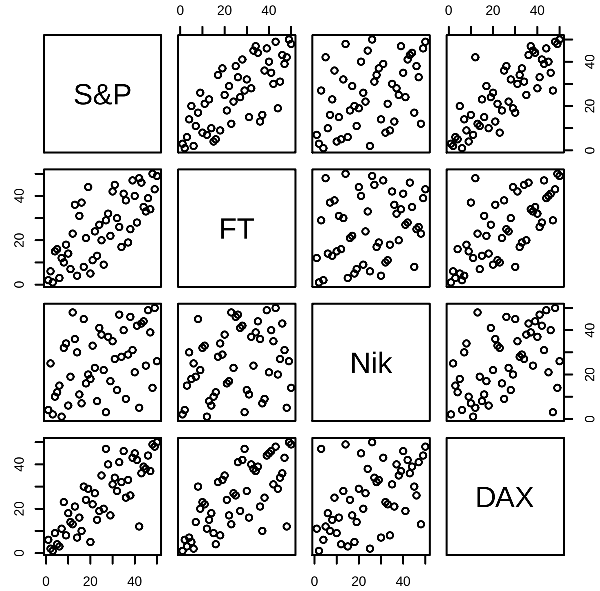

Figure 3. Scatter plot of all possible pairs of the market indices for the ranked data, Example 5.2

which is an extension of the correlation coefficient by [13] for more than two sets of rankings. The sampling period was September 2001-December 2005, yielding a sample size of $n = 50$ filtered returns (filtered by Meintanis and Iliopoulos [15] using ARMA(1,1) process).

We consider the 6 possible pairs of these market indices, (S&P, FT), (S&P, Nik), (S&P, DAX), (FT, Nik), (FT, DAX) and (Nik, DAX). Figure 3 shows the matrix of all possible pairs of market indices for the ranks of the original data (after filtering). We can see from this figure that the pair (S&P, DAX) (respectively, (FT, Nik)) is more (respectively, less) in rank agreement, which coincides with the results obtained by Meintanis and Iliopoulos [15]. Meintanis and Iliopoulos [15] found that all pairs of these four indices are also highly dependent, except for (FT, Nik), the most dependent pair was (S&P, DAX) and the least dependent pair was (FT, Nik) followed by (S&P, Nik).

Table 10 presents the Spearman's rank correlation coefficient R_s [18], Kendall's rank correlation R_k [10], the weighted rank correlation R_w as presented in (3) for different values of w, and the weighted correlation coefficients based on the order statistics and the quantiles of the Laplace distribution, R_O and R_L, respectively. In order to test the null hypothesis of independence against the alternative of a positive correlation, the critical values from the tables in Section 3, for significance level 5% and 1%, are used. Table 10 shows that for the pairs (FT, DAX) and (S&P, DAX) we always reject the null hypothesis of independence at significance level 1%, so there is evidence of positive correlation between these pairs of indices. Spearman's rank correlation coefficient R_s, Kendall's rank correlation R_k, and the two weighted correlation coefficients R_O and R_L always indicate positive correlation for all 6 pairs at significance level 1% (except for (FT, NiK) which is significant at 5%). However, for R_w this varies depending on the value of w. For example, we do not reject the null hypothesis of independence for the pair (FT, Nik) for $w = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$. We can also notice that the results from Spearman's rank correlation coefficient R_s are close to the results from the weighted rank correlation $R_{0.9}$, while the results from the two weighted correlation coefficients that based on the Laplace distribution, R_O and R_L are close to each other and close to the results from the weighted rank correlation $R_{0.9}$.

6. Concluding remarks

In this paper we have presented three new weighted rank correlation coefficients which are aimed of evaluating the agreement of the top and the bottom rankings simultaneously. In addition to the exact and approximate quantiles, we also derived the limiting distribution of these weighted rank coefficients under the null hypothesis of no agreement between the rankings. We illustrated the use of the new weighted rank coefficients via examples, including an example using financial markets indices.

We also carried out a simulation study to compare the performance of the three proposed weighted rank correlation coefficients with the well-known non-weighted rank correlation coefficients, namely Spearman's rho and Kendall's tau correlation coefficients. The simulation study showed that the proposed weighted correlation coefficients perform very well, compared with Spearman's rho and Kendall's tau correlation coefficients for $p = 0.1$, while their performances are close to Spearman's rho and Kendall's tau for $p = 0.3$. For p, σ and n, the R_w performs better than other correlation coefficients (except for large w), however, R_w performs worse for large σ. On the other hand, the Laplace distribution based rank correlation coefficients R_L and R_O perform very well.

In addition to the simulation study that has been presented in this paper, one can also apply other methods to compare these correlation measures, e.g. via resampling (bootstrap) techniques as in [14] or by visualising these concordance measures as in [22]. One can also consider extending the work presented in this paper to evaluate the agreement with focus on the top and the bottom rankings simultaneously when there are more than two rankings as in [3, 20], or even with focus on a specific range of rankings (e.g. measuring the agreement between nonlinear rankings [19]).

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Table 2. Exact quantiles for the weighted correlation, R_w

	w	90%	92.5%	95%	97.5%	99%	99.5%	99.9%
$n = 4$	0.1	0.5990	0.9802	0.9802	1.0000	1.0000	1.0000	1.0000
	0.2	0.6923	0.9231	0.9231	1.0000	1.0000	1.0000	1.0000
	0.3	0.7752	0.8349	0.8349	1.0000	1.0000	1.0000	1.0000
	0.4	0.8448	0.8448	0.8448	1.0000	1.0000	1.0000	1.0000
	0.5	0.9000	0.9000	0.9000	1.0000	1.0000	1.0000	1.0000
	0.6	0.9412	0.9412	0.9412	1.0000	1.0000	1.0000	1.0000
	0.7	0.9698	0.9698	0.9698	1.0000	1.0000	1.0000	1.0000
	0.8	0.9878	0.9878	0.9878	1.0000	1.0000	1.0000	1.0000
	0.9	0.9972	0.9972	0.9972	1.0000	1.0000	1.0000	1.0000
$n = 5$	0.1	0.5495	0.5941	0.5990	0.9851	0.9950	1.0000	1.0000
	0.2	0.5962	0.6731	0.6923	0.9423	0.9808	1.0000	1.0000
	0.3	0.6376	0.7339	0.7752	0.8761	0.9587	1.0000	1.0000
	0.4	0.6724	0.7241	0.7931	0.8448	0.9310	1.0000	1.0000
	0.5	0.7000	0.7000	0.8000	0.9000	0.9000	1.0000	1.0000
	0.6	0.6618	0.7206	0.8088	0.8824	0.9412	1.0000	1.0000
	0.7	0.7047	0.7349	0.8054	0.9396	0.9698	1.0000	1.0000
	0.8	0.7317	0.7439	0.7927	0.9756	0.9878	1.0000	1.0000
	0.9	0.7459	0.7486	0.7735	0.9945	0.9972	1.0000	1.0000
$n = 6$	0.1	0.5381	0.5449	0.5549	0.9851	0.9949	0.9960	1.0000
	0.2	0.5576	0.5791	0.6190	0.9416	0.9785	0.9877	1.0000
	0.3	0.5632	0.6029	0.6899	0.8734	0.9480	0.9799	1.0000
	0.4	0.5742	0.6309	0.7389	0.8192	0.9001	0.9568	1.0000
	0.5	0.5952	0.6905	0.7381	0.8095	0.8810	0.9524	1.0000
	0.6	0.5661	0.6627	0.7573	0.8625	0.9270	0.9613	1.0000
	0.7	0.5210	0.5800	0.7224	0.9248	0.9612	0.9745	1.0000
	0.8	0.4641	0.5168	0.6003	0.9684	0.9840	0.9902	1.0000
	0.9	0.4000	0.4308	0.4679	0.9927	0.9963	0.9980	1.0000
$n = 7$	0.1	0.5049	0.5400	0.5489	0.5989	0.9909	0.9949	0.9998
	0.2	0.5184	0.5599	0.5906	0.6905	0.9670	0.9785	0.9969
	0.3	0.5264	0.5669	0.6229	0.7658	0.9271	0.9480	0.9852
	0.4	0.5263	0.5796	0.6552	0.7625	0.8704	0.9109	0.9676
	0.5	0.5476	0.5952	0.6667	0.7619	0.8333	0.8810	0.9524
	0.6	0.5526	0.6090	0.6740	0.7653	0.8512	0.8888	0.9565
	0.7	0.5297	0.6020	0.6936	0.7832	0.8754	0.9170	0.9740
	0.8	0.5250	0.5640	0.7002	0.8123	0.8763	0.9586	0.9875
	0.9	0.5109	0.5330	0.6009	0.8285	0.8615	0.9907	0.9967
$n = 8$	0.1	0.4995	0.5060	0.5444	0.5544	0.9896	0.9944	0.9955
	0.2	0.4963	0.5276	0.5739	0.6143	0.9579	0.9739	0.9844
	0.3	0.4911	0.5373	0.5870	0.6727	0.9027	0.9398	0.9703
	0.4	0.4864	0.5379	0.6034	0.7152	0.8288	0.8884	0.9547
	0.5	0.5000	0.5471	0.6176	0.7059	0.7941	0.8471	0.9353
	0.6	0.5054	0.5569	0.6220	0.7112	0.7965	0.8399	0.9265
	0.7	0.5088	0.5558	0.6150	0.7125	0.8411	0.8903	0.9495
	0.8	0.5152	0.5506	0.5936	0.6615	0.9294	0.9519	0.9789
	0.9	0.5163	0.5297	0.5512	0.5913	0.9832	0.9886	0.9950
$n = 9$	0.1	0.4955	0.5004	0.5396	0.5494	0.9889	0.9905	0.9954
	0.2	0.4840	0.5029	0.5583	0.5944	0.9506	0.9646	0.9826
	0.3	0.4672	0.5075	0.5623	0.6299	0.8844	0.9261	0.9622
	0.4	0.4583	0.5069	0.5648	0.6615	0.8056	0.8709	0.9337
	0.5	0.4647	0.5118	0.5765	0.6647	0.7588	0.8176	0.9000
	0.6	0.4699	0.5201	0.5805	0.6658	0.7515	0.8010	0.8843
	0.7	0.5210	0.5800	0.7224	0.9248	0.9612	0.9745	1.0000
	0.8	0.4653	0.5164	0.5810	0.6741	0.8031	0.8560	0.9350
	0.9	0.4441	0.5131	0.5942	0.6558	0.8539	0.8772	0.9843
$n = 10$	0.1	0.4950	0.4994	0.5054	0.5450	0.9850	0.9901	0.9950
	0.2	0.4800	0.4952	0.5222	0.5804	0.9401	0.9610	0.9801
	0.3	0.4550	0.4867	0.5384	0.6068	0.8658	0.9140	0.9558
	0.4	0.4347	0.4805	0.5381	0.6235	0.7800	0.8491	0.9212
	0.5	0.4340	0.4809	0.5411	0.6320	0.7317	0.7918	0.8827
	0.6	0.4415	0.4883	0.5470	0.6302	0.7168	0.7675	0.8553
	0.7	0.4408	0.4907	0.5516	0.6372	0.7275	0.7804	0.8647
	0.8	0.4329	0.4963	0.5666	0.6456	0.7205	0.7807	0.9332
	0.9	0.4762	0.5531	0.5941	0.6361	0.6761	0.7103	0.9838
$n = 11$	0.1	0.4946	0.4957	0.5005	0.5445	0.5940	0.9900	0.9950
	0.2	0.4768	0.4860	0.5039	0.5761	0.6726	0.9592	0.9792
	0.3	0.4460	0.4720	0.5117	0.5922	0.7309	0.9066	0.9511
	0.4	0.4166	0.4593	0.5148	0.5968	0.7493	0.8332	0.9108
	0.5	0.4120	0.4560	0.5132	0.6012	0.7053	0.7698	0.8651
	0.6	0.4170	0.4621	0.5183	0.6000	0.6868	0.7395	0.8315
	0.7	0.4177	0.4640	0.5223	0.6050	0.6910	0.7421	0.8291
	0.8	0.4143	0.4630	0.5235	0.6104	0.7036	0.7671	0.8684
	0.9	0.4136	0.4733	0.5249	0.6109	0.7078	0.7564	0.9023
$n = 12$	0.1	0.4943	0.4954	0.5000	0.5440	0.5499	0.9895	0.9949
	0.2	0.4740	0.4824	0.4993	0.5728	0.5992	0.9568	0.9784
	0.3	0.4381	0.4627	0.4968	0.5819	0.6457	0.8997	0.9474
	0.4	0.4022	0.4424	0.4949	0.5773	0.6866	0.8193	0.9019
	0.5	0.3919	0.4355	0.4916	0.5766	0.6806	0.7498	0.8505
	0.6	0.3961	0.4395	0.4942	0.5741	0.6608	0.7148	0.8099
	0.7	0.3976	0.4420	0.4976	0.5774	0.6612	0.7122	0.8010
	0.8	0.3937	0.4377	0.4953	0.5856	0.6776	0.7290	0.8394
	0.9	0.3753	0.4026	0.4492	0.6303	0.6919	0.7206	0.9545

Table 3. Approximate quantiles for the weighted correlation, R_w (Cont.)

	w	90%	92.5%	95%	97.5%	99%	99.5%	99.9%
$n = 13$	0.1	0.4910	0.4950	0.4995	0.5400	0.5494	0.9894	0.9944
	0.2	0.4684	0.4803	0.4957	0.5645	0.5951	0.9552	0.9755
	0.3	0.4294	0.4566	0.4873	0.5659	0.6312	0.8894	0.9392
	0.4	0.3903	0.4294	0.4788	0.5613	0.6561	0.8052	0.8942
	0.5	0.3744	0.4168	0.4714	0.5542	0.6560	0.7289	0.8366
	0.6	0.3778	0.4198	0.4731	0.5511	0.6379	0.6931	0.7918
	0.7	0.3802	0.4223	0.4759	0.5532	0.6340	0.6833	0.7754
	0.8	0.3785	0.4221	0.4778	0.5590	0.6462	0.7020	0.7990
	0.9	0.3736	0.4163	0.4733	0.5667	0.6522	0.7162	0.8590
$n = 14$	0.1	0.4904	0.4950	0.4960	0.5098	0.5454	0.9851	0.9910
	0.2	0.4626	0.4794	0.4880	0.5227	0.5827	0.9408	0.9676
	0.3	0.4207	0.4526	0.4787	0.5450	0.6129	0.8692	0.9339
	0.4	0.3772	0.4181	0.4641	0.5426	0.6328	0.7846	0.8837
	0.5	0.3613	0.4032	0.4553	0.5369	0.6358	0.7134	0.8259
	0.6	0.3618	0.4026	0.4551	0.5321	0.6170	0.6734	0.7745
	0.7	0.3645	0.4060	0.4578	0.5331	0.6137	0.6641	0.7546
	0.8	0.3619	0.4043	0.4579	0.5359	0.6221	0.6775	0.7743
	0.9	0.3799	0.4180	0.4572	0.5147	0.6563	0.7127	0.7766
$n = 15$	0.1	0.4900	0.4949	0.4955	0.5009	0.5450	0.5940	0.9905
	0.2	0.4606	0.4783	0.4843	0.5069	0.5795	0.6726	0.9645
	0.3	0.4132	0.4485	0.4688	0.5203	0.6027	0.7317	0.9252
	0.4	0.3677	0.4113	0.4535	0.5245	0.6143	0.7548	0.8749
	0.5	0.3487	0.3902	0.4417	0.5208	0.6177	0.6979	0.8172
	0.6	0.3473	0.3871	0.4381	0.5143	0.5996	0.6558	0.7564
	0.7	0.3506	0.3904	0.4400	0.5129	0.5910	0.6399	0.7332
	0.8	0.3497	0.3901	0.4415	0.5171	0.5989	0.6517	0.7468
	0.9	0.3487	0.3895	0.4404	0.5182	0.6105	0.6676	0.7734
$n = 16$	0.1	0.4896	0.4946	0.4955	0.5004	0.5446	0.5500	0.9905
	0.2	0.4571	0.4767	0.4831	0.5023	0.5768	0.5990	0.9638
	0.3	0.4072	0.4457	0.4656	0.5071	0.5963	0.6467	0.9221
	0.4	0.3573	0.4042	0.4433	0.5091	0.6012	0.6840	0.8688
	0.5	0.3377	0.3785	0.4282	0.5057	0.5981	0.6761	0.8046
	0.6	0.3352	0.3741	0.4239	0.4993	0.5831	0.6407	0.7461
	0.7	0.3381	0.3763	0.4251	0.4961	0.5728	0.6226	0.7145
	0.8	0.3370	0.3761	0.4265	0.5003	0.5804	0.6308	0.7241
	0.9	0.3254	0.3668	0.4321	0.5094	0.5767	0.6385	0.7615
$n = 17$	0.1	0.4851	0.4945	0.4951	0.5000	0.5445	0.5495	0.9905
	0.2	0.4410	0.4761	0.4814	0.4999	0.5762	0.5953	0.9632
	0.3	0.3800	0.4425	0.4608	0.4988	0.5927	0.6335	0.9205
	0.4	0.3465	0.3980	0.4358	0.4977	0.5919	0.6577	0.8615
	0.5	0.3273	0.3690	0.4172	0.4919	0.5832	0.6600	0.7951
	0.6	0.3242	0.3627	0.4119	0.4859	0.5719	0.6294	0.7393
	0.7	0.3274	0.3649	0.4120	0.4817	0.5577	0.6064	0.6988
	0.8	0.3259	0.3643	0.4130	0.4844	0.5624	0.6120	0.7061
	0.9	0.3237	0.3626	0.4128	0.4870	0.5700	0.6242	0.7267
$n = 18$	0.1	0.4455	0.4945	0.4950	0.4999	0.5445	0.5490	0.9901
	0.2	0.3849	0.4754	0.4806	0.4991	0.5758	0.5913	0.9618
	0.3	0.3283	0.4391	0.4581	0.4957	0.5908	0.6238	0.9158
	0.4	0.3351	0.3919	0.4296	0.4882	0.5856	0.6414	0.8551
	0.5	0.3171	0.3601	0.4069	0.4798	0.5720	0.6412	0.7860
	0.6	0.3135	0.3511	0.3996	0.4727	0.5566	0.6137	0.7207
	0.7	0.3158	0.3522	0.3985	0.4668	0.5427	0.5921	0.6847
	0.8	0.3161	0.3528	0.4003	0.4694	0.5455	0.5941	0.6836
	0.9	0.3170	0.3522	0.3970	0.4721	0.5581	0.6049	0.7170
$n = 19$	0.1	0.4450	0.4940	0.4950	0.4999	0.5445	0.5450	0.9901
	0.2	0.3801	0.4723	0.4802	0.4984	0.5753	0.5806	0.9609
	0.3	0.3138	0.4307	0.4558	0.4918	0.5878	0.6065	0.9139
	0.4	0.3216	0.3847	0.4244	0.4808	0.5794	0.6259	0.8523
	0.5	0.3080	0.3527	0.3995	0.4695	0.5620	0.6269	0.7794
	0.6	0.3045	0.3417	0.3890	0.4602	0.5438	0.6008	0.7087
	0.7	0.3071	0.3428	0.3878	0.4550	0.5296	0.5789	0.6707
	0.8	0.3080	0.3438	0.3894	0.4565	0.5300	0.5769	0.6647
	0.9	0.3060	0.3424	0.3891	0.4599	0.5400	0.5913	0.6921
$n = 20$	0.1	0.0995	0.4905	0.4950	0.4995	0.5440	0.5450	0.9900
	0.2	0.1975	0.4646	0.4800	0.4954	0.5730	0.5798	0.9606
	0.3	0.2817	0.4252	0.4550	0.4848	0.5825	0.6027	0.9117
	0.4	0.3034	0.3788	0.4216	0.4745	0.5723	0.6149	0.8478
	0.5	0.2975	0.3455	0.3916	0.4605	0.5533	0.6124	0.7697
	0.6	0.2958	0.3325	0.3790	0.4499	0.5318	0.5884	0.7018
	0.7	0.2979	0.3329	0.3773	0.4431	0.5170	0.5654	0.6586
	0.8	0.2989	0.3339	0.3783	0.4442	0.5163	0.5628	0.6519
	0.9	0.2955	0.3322	0.3786	0.4462	0.5225	0.5762	0.6717
$n = 30$	0.1	0.0496	0.0545	0.4945	0.4950	0.4999	0.5440	0.9895
	0.2	0.0975	0.1160	0.4762	0.4807	0.4992	0.5722	0.9562
	0.3	0.1433	0.1813	0.4426	0.4579	0.4952	0.5783	0.8979
	0.4	0.1847	0.2410	0.3933	0.4288	0.4859	0.5613	0.8108
	0.5	0.2171	0.2730	0.3492	0.3989	0.4695	0.5406	0.7161
	0.6	0.2341	0.2704	0.3155	0.3780	0.4531	0.5090	0.6299
	0.7	0.2379	0.2678	0.3065	0.3658	0.4350	0.4821	0.5753
	0.8	0.2403	0.2688	0.3055	0.3606	0.4231	0.4644	0.5445
	0.9	0.2399	0.2689	0.3066	0.3617	0.4249	0.4667	0.5481

Table 4. Approximate quantiles for the weighted correlation, R_w (Cont.)

	w	90%	92.5%	95%	97.5%	99%	99.5%	99.9%
$n = 40$	0.1	0.0495	0.0495	0.0990	0.4950	0.4955	0.5000	0.5450
	0.2	0.0960	0.0968	0.1928	0.4800	0.4838	0.4992	0.5783
	0.3	0.1362	0.1402	0.2767	0.4550	0.4673	0.4961	0.6035
	0.4	0.1667	0.1794	0.3218	0.4201	0.4470	0.4879	0.6030
	0.5	0.1856	0.2099	0.2888	0.3774	0.4252	0.4745	0.6086
	0.6	0.1970	0.2300	0.2803	0.3404	0.4062	0.4570	0.5765
	0.7	0.2036	0.2311	0.2671	0.3219	0.3870	0.4315	0.5255
	0.8	0.2065	0.2315	0.2640	0.3135	0.3704	0.4087	0.4840
	0.9	0.2066	0.2316	0.2636	0.3115	0.3674	0.4037	0.4765
$n = 50$	0.1	0.0490	0.0495	0.0500	0.4950	0.4950	0.4955	0.5445
	0.2	0.0922	0.0960	0.0999	0.4799	0.4808	0.4846	0.5760
	0.3	0.1254	0.1365	0.1491	0.4545	0.4587	0.4746	0.5918
	0.4	0.1456	0.1681	0.1956	0.4175	0.4308	0.4586	0.5888
	0.5	0.1643	0.1891	0.2342	0.3710	0.4008	0.4462	0.5675
	0.6	0.1747	0.2020	0.2484	0.3198	0.3767	0.4231	0.5315
	0.7	0.1785	0.2048	0.2395	0.2913	0.3522	0.3963	0.4875
	0.8	0.1823	0.2052	0.2348	0.2805	0.3339	0.3693	0.4436
	0.9	0.1847	0.2067	0.2355	0.2786	0.3290	0.3617	0.4259
$n = 60$	0.1	0.0054	0.0495	0.0495	0.4950	0.4950	0.4955	0.5445
	0.2	0.0230	0.0959	0.0962	0.4793	0.4802	0.4838	0.5760
	0.3	0.0533	0.1357	0.1377	0.4516	0.4561	0.4672	0.5914
	0.4	0.0942	0.1654	0.1733	0.4121	0.4244	0.4467	0.5869
	0.5	0.1287	0.1820	0.2006	0.3628	0.3874	0.4218	0.5608
	0.6	0.1509	0.1859	0.2213	0.3080	0.3564	0.3983	0.5118
	0.7	0.1620	0.1869	0.2204	0.2705	0.3286	0.3718	0.4604
	0.8	0.1664	0.1876	0.2155	0.2581	0.3085	0.3426	0.4149
	0.9	0.1676	0.1880	0.2142	0.2541	0.2995	0.3303	0.3922
$n = 70$	0.1	0.0050	0.0490	0.0495	0.4945	0.4950	0.4950	0.5049
	0.2	0.0193	0.0922	0.0960	0.4762	0.4800	0.4808	0.5184
	0.3	0.0420	0.1253	0.1366	0.4427	0.4553	0.4588	0.5372
	0.4	0.0706	0.1445	0.1685	0.3931	0.4215	0.4316	0.5540
	0.5	0.1018	0.1641	0.1902	0.3460	0.3806	0.4030	0.5155
	0.6	0.1313	0.1707	0.2026	0.2951	0.3404	0.3815	0.4957
	0.7	0.1475	0.1724	0.2048	0.2557	0.3095	0.3505	0.4356
	0.8	0.1524	0.1727	0.1993	0.2405	0.2887	0.3226	0.3923
	0.9	0.1547	0.1734	0.1980	0.2349	0.2769	0.3057	0.3633
$n = 80$	0.1	0.0050	0.0054	0.0495	0.0995	0.4950	0.4950	0.5000
	0.2	0.0192	0.0230	0.0960	0.1926	0.4800	0.4807	0.4993
	0.3	0.0410	0.0532	0.1365	0.2732	0.4550	0.4584	0.4963
	0.4	0.0673	0.0948	0.1680	0.3351	0.4204	0.4300	0.4922
	0.5	0.0943	0.1294	0.1874	0.2826	0.3770	0.3960	0.4801
	0.6	0.1187	0.1520	0.1930	0.2752	0.3304	0.3645	0.4610
	0.7	0.1357	0.1609	0.1917	0.2445	0.2951	0.3343	0.4202
	0.8	0.1419	0.1610	0.1862	0.2256	0.2730	0.3062	0.3738
	0.9	0.1447	0.1625	0.1854	0.2204	0.2606	0.2884	0.3441
$n = 90$	0.1	0.0049	0.0050	0.0495	0.0505	0.4950	0.4950	0.5000
	0.2	0.0192	0.0194	0.0960	0.0999	0.4800	0.4802	0.4992
	0.3	0.0409	0.0421	0.1364	0.1491	0.4550	0.4561	0.4960
	0.4	0.0669	0.0715	0.1673	0.1991	0.4200	0.4243	0.4873
	0.5	0.0925	0.1044	0.1857	0.2360	0.3755	0.3868	0.4692
	0.6	0.1123	0.1341	0.1877	0.2507	0.3241	0.3523	0.4409
	0.7	0.1262	0.1503	0.1820	0.2348	0.2848	0.3215	0.4076
	0.8	0.1331	0.1516	0.1759	0.2139	0.2595	0.2922	0.3588
	0.9	0.1362	0.1528	0.1746	0.2076	0.2460	0.2716	0.3241
$n = 100$	0.1	0.0049	0.0050	0.0495	0.0495	0.4950	0.4950	0.4999
	0.2	0.0191	0.0192	0.0958	0.0968	0.4800	0.4800	0.4992
	0.3	0.0399	0.0410	0.1354	0.1402	0.4550	0.4553	0.4958
	0.4	0.0635	0.0678	0.1640	0.1788	0.4200	0.4220	0.4871
	0.5	0.0879	0.0963	0.1816	0.2111	0.3750	0.3820	0.4682
	0.6	0.1048	0.1224	0.1810	0.2322	0.3212	0.3442	0.4333
	0.7	0.1187	0.1408	0.1744	0.2241	0.2741	0.3093	0.3939
	0.8	0.1256	0.1435	0.1674	0.2043	0.2488	0.2798	0.3453
	0.9	0.1289	0.1447	0.1653	0.1969	0.2336	0.2584	0.3102

Table 5. Exact and approximate quantiles R_L

n	90%	92.5%	95%	97.5%	99%	99.5%	99.9%
4	0.7299	0.8880	0.8880	1.0000	1.0000	1.0000	1.0000
5	0.6624	0.7602	0.8202	0.8248	0.9401	1.0000	1.0000
6	0.5804	0.6434	0.7328	0.8233	0.8992	0.9569	1.0000
7	0.5467	0.5967	0.6603	0.7499	0.8466	0.8935	0.9503
8	0.5021	0.5510	0.6130	0.7055	0.7932	0.8439	0.9225
9	0.4687	0.5180	0.5771	0.6641	0.7534	0.8047	0.8888
10	0.4412	0.4878	0.5460	0.6292	0.7168	0.7693	0.8588
11	0.4175	0.4625	0.5191	0.5998	0.6858	0.7385	0.8301
12	0.3973	0.4408	0.4953	0.5742	0.6583	0.7105	0.8031
13	0.3794	0.4215	0.4740	0.5510	0.6339	0.6855	0.7812
14	0.3646	0.4057	0.4571	0.5320	0.6139	0.6645	0.7566
15	0.3500	0.3900	0.4401	0.5132	0.5926	0.6435	0.7347
16	0.3389	0.3779	0.4265	0.4976	0.5752	0.6253	0.7163
17	0.3268	0.3643	0.4121	0.4818	0.5582	0.6067	0.6985
18	0.3169	0.3533	0.3999	0.4687	0.5432	0.5916	0.6813
19	0.3077	0.3434	0.3889	0.4559	0.5292	0.5759	0.6660
20	0.2981	0.3331	0.3772	0.4432	0.5160	0.5625	0.6509
30	0.2404	0.2694	0.3064	0.3614	0.4237	0.4647	0.5445
40	0.2067	0.2316	0.2640	0.3125	0.3678	0.4052	0.4775
50	0.1847	0.2070	0.2359	0.2798	0.3294	0.3621	0.4298
60	0.1677	0.1882	0.2144	0.2544	0.3004	0.3311	0.3909
70	0.1544	0.1732	0.1978	0.2352	0.2780	0.3068	0.3656
80	0.1445	0.1623	0.1852	0.2202	0.2600	0.2867	0.3416
90	0.1360	0.1526	0.1742	0.2070	0.2444	0.2698	0.3210
100	0.1291	0.1449	0.1656	0.1971	0.2333	0.2581	0.3065

Table 6. Exact and approximate quantiles for R_O

n	90%	92.5%	95%	97.5%	99%	99.5%	99.9%
4	0.7337	0.8840	0.8840	1.0000	1.0000	1.0000	1.0000
5	0.6596	0.7616	0.8191	0.8273	0.9424	1.0000	1.0000
6	0.5872	0.6355	0.7260	0.8248	0.9045	0.9618	1.0000
7	0.5405	0.5957	0.6623	0.7533	0.8475	0.8898	0.9549
8	0.4999	0.5497	0.6123	0.7076	0.7984	0.8509	0.9282
9	0.4668	0.5158	0.5756	0.6653	0.7545	0.8092	0.8953
10	0.4394	0.4861	0.5448	0.6301	0.7195	0.7734	0.8635
11	0.4161	0.4612	0.5177	0.6001	0.6882	0.7420	0.8351
12	0.3962	0.4396	0.4943	0.5742	0.6605	0.7140	0.8085
13	0.3786	0.4206	0.4734	0.5510	0.6375	0.6907	0.7871
14	0.3632	0.4040	0.4557	0.5316	0.6138	0.6654	0.7632
15	0.3498	0.3892	0.4393	0.5130	0.5937	0.6455	0.7425
16	0.3379	0.3769	0.4252	0.4965	0.5755	0.6259	0.7212
17	0.3263	0.3639	0.4113	0.4812	0.5586	0.6091	0.7035
18	0.3160	0.3527	0.3992	0.4675	0.5438	0.5926	0.6877
19	0.3068	0.3426	0.3878	0.4554	0.5300	0.5780	0.6707
20	0.2991	0.3338	0.3781	0.4439	0.5167	0.5634	0.6562
30	0.2404	0.2691	0.3059	0.3610	0.4235	0.4652	0.5479
40	0.2066	0.2316	0.2636	0.3116	0.3676	0.4050	0.4790
50	0.1841	0.2065	0.2354	0.2795	0.3301	0.3642	0.4319
60	0.1675	0.1880	0.2146	0.2547	0.3014	0.3319	0.3947
70	0.1548	0.1736	0.1981	0.2352	0.2785	0.3076	0.3667
80	0.1444	0.1620	0.1849	0.2196	0.2601	0.2878	0.3445
90	0.1363	0.1529	0.1746	0.2074	0.2459	0.2711	0.3231
100	0.1290	0.1450	0.1656	0.1967	0.2324	0.2569	0.3077

Table 7. Power comparisons of R_k, R_s, R_w, R_L and R_O

	n	σ	R_k	R_s	$R_{0.4}$	$R_{0.5}$	$R_{0.6}$	$R_{0.7}$	$R_{0.8}$	R_L	R_O
$p = 0.1$	10	0.25	0.3496	0.3693	0.9723	0.9615	0.7200	0.4312	0.2518	0.7617	0.8471
		0.50	0.3023	0.3211	0.8153	0.7974	0.6049	0.3785	0.2339	0.6358	0.7033
		1	0.1961	0.2203	0.4638	0.4478	0.3573	0.2496	0.1793	0.3727	0.4037
		2	0.1092	0.1329	0.2011	0.2001	0.1745	0.1433	0.1249	0.1777	0.1883
		3	0.0839	0.1027	0.1409	0.1409	0.1282	0.1110	0.1073	0.1289	0.1344
	20	0.25	0.7857	0.7834	1.0000	1.0000	1.0000	0.9999	0.9833	0.9999	0.9999
		0.50	0.7367	0.7287	0.9903	0.9951	0.9940	0.9898	0.9351	0.9818	0.9865
		1	0.4973	0.4842	0.7754	0.8084	0.8144	0.7852	0.6401	0.7453	0.7663
		2	0.2461	0.2402	0.3851	0.3912	0.3924	0.3688	0.2953	0.3444	0.3561
		3	0.1683	0.1654	0.2643	0.2516	0.2425	0.2346	0.1964	0.2226	0.2294
	30	0.25	0.9404	0.9545	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
		0.50	0.9084	0.9223	0.9963	0.9998	1.0000	1.0000	0.9999	0.9994	0.9997
		1	0.6871	0.6964	0.8123	0.8842	0.9253	0.9432	0.9245	0.9161	0.9242
		2	0.3332	0.3417	0.4020	0.4569	0.5128	0.5378	0.5083	0.4993	0.5106
		3	0.2106	0.2165	0.2478	0.2709	0.2989	0.3164	0.3002	0.2951	0.3005
	50	0.25	0.9985	0.9994	0.9996	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
		0.50	0.9952	0.9973	0.9935	0.9998	1.0000	1.0000	1.0000	1.0000	1.0000
		1	0.9098	0.9117	0.8272	0.9135	0.9655	0.9892	0.9959	0.9933	0.9942
		2	0.5345	0.5285	0.3758	0.4655	0.5734	0.6789	0.7495	0.7212	0.7319
		3	0.3167	0.3124	0.2217	0.2718	0.3403	0.4073	0.4625	0.4405	0.4445
	100	0.25	1.0000	1.0000	0.9995	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
		0.50	1.0000	1.0000	0.9820	0.9979	0.9999	1.0000	1.0000	1.0000	1.0000
		1	0.9976	0.9979	0.8128	0.9014	0.9683	0.9964	0.9999	1.0000	1.0000
		2	0.7971	0.7967	0.4107	0.4591	0.5675	0.7228	0.8839	0.9479	0.9486
		3	0.5281	0.5281	0.2557	0.2807	0.3359	0.4443	0.6054	0.7140	0.7168
$p = 0.2$	10	0.25	0.9161	0.9571	0.9859	0.9885	0.9867	0.9585	0.6764	0.9882	0.9883
		0.50	0.7705	0.8152	0.9045	0.9200	0.9091	0.8270	0.5575	0.9158	0.9168
		1	0.4647	0.5070	0.5768	0.6066	0.5955	0.5208	0.3753	0.6071	0.6073
		2	0.1999	0.2344	0.2613	0.2763	0.2725	0.2441	0.2134	0.2741	0.2758
		3	0.1236	0.1528	0.1637	0.1676	0.1669	0.1599	0.1515	0.1695	0.1686
	20	0.25	0.9998	1.0000	0.9986	0.9998	1.0000	1.0000	1.0000	1.0000	1.0000
		0.50	0.9959	0.9967	0.9693	0.9940	0.9988	0.9994	0.9990	0.9993	0.9993
		1	0.8610	0.8620	0.7456	0.8335	0.9027	0.9305	0.9059	0.9290	0.9271
		2	0.4419	0.4407	0.3782	0.4241	0.4685	0.5142	0.4884	0.5123	0.5129
		3	0.2691	0.2699	0.2652	0.2748	0.2930	0.3141	0.2921	0.3095	0.3101
	30	0.25	1.0000	1.0000	0.9986	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000
		0.50	0.9999	1.0000	0.9689	0.9948	0.9997	1.0000	1.0000	1.0000	1.0000
		1	0.9728	0.9748	0.7429	0.8525	0.9418	0.9833	0.9919	0.9918	0.9912
		2	0.6087	0.6246	0.3665	0.4381	0.5388	0.6488	0.7095	0.7048	0.7017
		3	0.3563	0.3699	0.2400	0.2733	0.3287	0.3958	0.4324	0.4272	0.4244
	50	0.25	1.0000	1.0000	0.9992	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
		0.50	1.0000	1.0000	0.9733	0.9938	0.9999	1.0000	1.0000	1.0000	1.0000
		1	0.9990	0.9992	0.7426	0.8574	0.9454	0.9906	0.9994	0.9997	0.9997
		2	0.8280	0.8274	0.3399	0.4332	0.5560	0.7077	0.8490	0.8891	0.8853
		3	0.5462	0.5459	0.2144	0.2623	0.3399	0.4478	0.5702	0.6221	0.6166
	100	0.25	1.0000	1.0000	0.9993	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
		0.50	1.0000	1.0000	0.9723	0.9934	0.9990	1.0000	1.0000	1.0000	1.0000
		1	1.0000	1.0000	0.7620	0.8392	0.9317	0.9884	0.9997	1.0000	1.0000
		2	0.9807	0.9807	0.3884	0.4327	0.5280	0.6859	0.8891	0.9929	0.9919
		3	0.8141	0.8164	0.2376	0.2605	0.3149	0.4230	0.6234	0.8830	0.8783
$p = 0.3$	10	0.25	0.9870	0.9939	0.9869	0.9935	0.9937	0.9923	0.9586	0.9947	0.9941
		0.50	0.9136	0.9386	0.8994	0.9392	0.9507	0.9325	0.8346	0.9557	0.9521
		1	0.6002	0.6570	0.6012	0.6603	0.6871	0.6527	0.5384	0.6948	0.6855
		2	0.2517	0.2944	0.2682	0.2941	0.3113	0.2970	0.2664	0.3148	0.3092
		3	0.1531	0.1894	0.1686	0.1824	0.1940	0.1924	0.1794	0.1954	0.1903
	20	0.25	1.0000	1.0000	0.9971	0.9995	1.0000	1.0000	1.0000	1.0000	1.0000
		0.50	0.9997	0.9997	0.9538	0.9881	0.9984	1.0000	0.9999	1.0000	1.0000
		1	0.9386	0.9417	0.7177	0.8116	0.9057	0.9529	0.9540	0.9588	0.9559
		2	0.5504	0.5543	0.3735	0.4248	0.4940	0.5711	0.5783	0.5904	0.5823
		3	0.3338	0.3351	0.2534	0.2709	0.3000	0.3441	0.3522	0.3544	0.3470
	30	0.25	1.0000	1.0000	0.9980	0.9998	1.0000	1.0000	1.0000	1.0000	1.0000
		0.50	1.0000	1.0000	0.9589	0.9901	0.9992	1.0000	1.0000	1.0000	1.0000
		1	0.9915	0.9927	0.7012	0.8249	0.9266	0.9851	0.9957	0.9961	0.9958
		2	0.7163	0.7356	0.3621	0.4355	0.5456	0.6685	0.7604	0.7653	0.7567
		3	0.4423	0.4572	0.2366	0.2706	0.3298	0.4100	0.4804	0.4849	0.4763
	50	0.25	1.0000	1.0000	0.9994	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
		0.50	1.0000	1.0000	0.9670	0.9912	0.9991	1.0000	1.0000	1.0000	1.0000
		1	0.9999	0.9999	0.7113	0.8248	0.9235	0.9849	0.9994	1.0000	0.9999
		2	0.9065	0.9102	0.3386	0.4309	0.5459	0.7023	0.8681	0.9318	0.9244
		3	0.6501	0.6483	0.2077	0.2578	0.3346	0.4407	0.5900	0.6836	0.6752
	100	0.25	1.0000	1.0000	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
		0.50	1.0000	1.0000	0.9691	0.9915	0.9991	1.0000	1.0000	1.0000	1.0000
		1	1.0000	1.0000	0.7388	0.8155	0.9170	0.9795	0.9994	1.0000	1.0000
		2	0.9955	0.9949	0.3853	0.4244	0.5137	0.6660	0.8766	0.9976	0.9972
		3	0.8961	0.8971	0.2419	0.2623	0.3149	0.4261	0.6183	0.9188	0.9151

Table 8. Data set, Example 5.1

A	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
B	1	2	3	4	11	10	9	8	7	6	5	12	13	14	15
C	1	2	3	4	10	8	6	11	7	5	9	12	13	14	15
D	1	2	8	12	10	7	6	5	3	13	9	4	11	14	15
E	15	14	13	12	5	6	7	8	9	10	11	4	3	2	1
F	2	1	3	4	5	6	7	8	9	10	11	12	13	15	14
G	3	2	1	4	5	6	7	8	9	10	11	12	15	14	13
H	4	3	2	1	5	6	7	8	9	10	11	15	14	13	12

Table 9. Rank correlation coefficients' results, Example 5.1

	(A,B)	(A,C)	(A,D)	(A,E)	(A,F)	(A,G)	(A,H)
R_s	0.8000**	0.8714**	0.5679*	-0.8000**	0.9929**	0.9714**	0.9286**
R_k	0.6000**	0.7524**	0.3905*	-0.6000**	0.9619**	0.8857**	0.7714**
$R_{0.1}$	1.0000**	1.0000**	0.9999**	-1.0000**	0.1981	0.0297	0.0040
$R_{0.2}$	1.0000**	1.0000**	0.9984**	-1.0000**	0.3856	0.1153	0.0307
$R_{0.3}$	0.9999**	0.9999**	0.9916**	-0.9999**	0.5541	0.2464	0.0983
$R_{0.4}$	0.9987**	0.9992**	0.9731**	-0.9987**	0.6976**	0.4073	0.2157
$R_{0.5}$	0.9923**	0.9951**	0.9332**	-0.9923**	0.8125**	0.5781*	0.3789
$R_{0.6}$	0.9679**	0.9799**	0.8608**	-0.9679**	0.8975**	0.7377**	0.5694*
$R_{0.7}$	0.8976**	0.9368**	0.7483**	-0.8976**	0.9538**	0.8664**	0.7557**
$R_{0.8}$	0.7410**	0.8434**	0.6034**	-0.7410**	0.9849**	0.9512**	0.9007**
$R_{0.9}$	0.4769*	0.6898**	0.4589*	-0.4769**	0.9975**	0.9911**	0.9799**
R_L	0.9197**	0.9482**	0.7583**	-0.9197**	0.9400**	0.8493**	0.7395**
R_O	0.9271**	0.9531**	0.7857**	-0.9271**	0.9157**	0.8102**	0.6953**

'*' significance at 5% and '**' significance at 1%

Table 10. Rank correlation coefficients for all pairs of the market indices, Example 5.2

	(S&P, FT)	(S&P, Nik)	(S&P, DAX)	(FT, Nik)	(FT, DAX)	(Nik, DAX)
R_s	0.7419**	0.4144**	0.8240**	0.2449*	0.6959**	0.4779**
R_k	0.5527**	0.2980**	0.6376**	0.1771*	0.5200**	0.3437**
$R_{0.1}$	0.1089*	0.0505*	0.5059**	0.0495	0.5941**	0.0544*
$R_{0.2}$	0.2304*	0.1039*	0.5262**	0.0967	0.6728**	0.1151*
$R_{0.3}$	0.3554*	0.1625*	0.5628**	0.1400	0.7324**	0.1765*
$R_{0.4}$	0.4731**	0.2278*	0.6139**	0.1778	0.7707**	0.2319*
$R_{0.5}$	0.5734**	0.2990*	0.6744**	0.2083	0.7885**	0.2744*
$R_{0.6}$	0.6476**	0.3713*	0.7352**	0.2296	0.7894**	0.2999*
$R_{0.7}$	0.6929**	0.4344**	0.7840**	0.2401	0.7791**	0.3130*
$R_{0.8}$	0.7212**	0.4737**	0.8134**	0.2439*	0.7619**	0.3408**
$R_{0.9}$	0.7572**	0.4568**	0.8371**	0.2560*	0.7379**	0.4428**
R_L	0.7616**	0.4660**	0.8338**	0.2646*	0.7558**	0.4200**
R_O	0.7602**	0.4656**	0.8317**	0.2679*	0.7652**	0.4161**

'*' significance at 5% and '**' significance at 1%