SURVIVAL SIGNATURE FOR SYSTEM RELIABILITY[1]

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1 Survival Signature

Coolen and Coolen-Maturi [6] introduced the survival signature for quantification of system reliability. Consider a system with $K \geq 1$ types of components, with n_k components of type $k \in \{1, 2, \ldots, K\}$ and $\sum_{k=1}^{K} n_k = n$. The essential assumption is that the random failure times of components of the same type are exchangeable [8, 12]. The state vector $\underline{x} \in \{0, 1\}^n$ of the system describes the states of its components, with 1 representing that a component functions and 0 that it does not function. The system structure function $\phi(\underline{x}) \in \{0, 1\}$ describes the functioning of the system given the component states $\underline{x}$, where 1 represents that the system functions and 0 that it does not function. Due to the arbitrary ordering of the components in the state vector, components of the same type can be grouped together, leading to a state vector that can be written as $\underline{x} = (\underline{x}^1, \underline{x}^2, \ldots, \underline{x}^K)$, with $\underline{x}^k = (x_1^k, x_2^k, \ldots, x_{n_k}^k)$ the sub-vector representing the states of the components of type k.

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The *survival signature*, denoted by $\Phi(l_1, l_2, \ldots, l_K)$, with $l_k = 0, 1, \ldots, n_k$ for $k = 1, \ldots, K$, is defined as the probability that the system functions given that *precisely* l_k of its n_k components of type k function, for each $k \in \{1, 2, \ldots, K\}$. There are $\binom{n_k}{l_k}$ state vectors $\underline{x}^k$ with $\sum_{i=1}^{n_k} x_i^k = l_k$; let S_l^k denote the set of these state vectors for components of type k and let $S_{l_1,\ldots,l_K}$ denote the set of all state vectors for the whole system for which $\sum_{i=1}^{n_k} x_i^k = l_k$, $k = 1, 2, \ldots, K$. Due to the exchangeability assumption for the failure times of the n_k components of type k, all the state vectors $\underline{x}^k \in S_l^k$ are equally likely to occur, hence

$$\Phi(l_1, \ldots, l_K) = \left[\prod_{k=1}^{K} \binom{n_k}{l_k}^{-1}\right] \times \sum_{\underline{x} \in S_{l_1,\ldots,l_K}} \phi(\underline{x}) \tag{1}$$

The survival signature is useful for deriving the probability for the event that the system functions at time $t > 0$, so for $T_S > t$, where T_S is the random system failure time. Let $C_k(t) \in \{0, 1, \ldots, n_k\}$ denote the number of components of type k in the system which function at time $t > 0$, then

$$P(T_S > t) = \sum_{l_1=0}^{n_1} \cdots \sum_{l_K=0}^{n_K} \left\{\Phi(l_1, \ldots, l_K) P\left(\bigcap_{k=1}^{K} \{C_k(t) = l_k\}\right)\right\} \tag{2}$$

Equation (2) is the essential result in survival signature theory. It shows that the system survival function can be computed with the required inputs, namely the information about the system structure and about the component failure times, being completely separated. Hence, the effect of changing a system's structure on its survival function can easily be investigated. One can also compare different system structures in general, without assumptions for the random failure times, by comparing the systems' survival signatures [28]. The system survival function is sufficient for important metrics such as the expected failure time of the system, or its remaining time till failure once it has been functioning for some time.

The survival signature requires specification at $\prod_{k=1}^{K}(n_k + 1)$ inputs while the structure function must be specified at 2^n inputs; in particular for large values of n and relatively small values of K, so large systems with few component types, the difference is enormous. If all components are of different types, so $K = n$, then the survival signature does not provide any advantages, in the sense of reduced representation, over the structure function. If all components are of the same type, so $K = 1$, then the survival function is closely related to Samaniego's system signature [26, 27].

Equation (2) only requires the assumption that failure times of components of the same type are exchangeable. If one assumes that the failure times of components of different types are independent, then Equation (2) becomes

$$P(T_S > t) = \sum_{l_1=0}^{n_1} \cdots \sum_{l_K=0}^{n_K} \left\{\Phi(l_1, \ldots, l_K) \prod_{k=1}^{K} P(C_k(t) = l_k)\right\} \tag{3}$$

If, in addition, one assumes that the failure times of components of the same type are independent and identically distributed (*iid*), with known cumulative distribution function (CDF) $F_k(t)$ for type k, then this leads to

$$P(T_S > t) = \sum_{l_1=0}^{n_1} \cdots \sum_{l_K=0}^{n_K} \left\{ \Phi(l_1, \ldots, l_K) \prod_{k=1}^{K} \binom{n_k}{l_k} [F_k(t)]^{n_k - l_k} [1 - F_k(t)]^{l_k} \right\} \quad (4)$$

One can also assume a parametric CDF to enable learning about the parameter based on data, e.g. using Bayesian statistics [2], or use a frequentist statistical method, for example Nonparametric Predictive Inference [9, 10]. The general formula for the system survival function, Equation (2), can also be applied if components' failure times are dependent, for example there may be common-cause failure modes, a risk of cascading failures, load sharing between components and so on. Initial studies into several of such possibilities have been published [7, 14, 15] and there are many related research challenges.

The survival signature has also been presented for multi-state systems with multi-state components, which enables application to a wide variety of practical problems, for example when components or systems can deteriorate and decisions about inspection and maintenance are required [23]. This opens up a wide range of research topics with focus on large-scale systems and networks.

2 Computational Aspects

For reliability of small systems and networks one can simply derive the system structure function and use Equation (1) to compute the survival signature. This approach has been implemented in the statistical software R [1], and can be used for small to medium-sized systems and networks. Reed et al [24] presented a substantial improvement on the required computation time by using binary decision diagrams, which can also be used for reliability of multi-terminal networks [25]. Using basic combinatorics, one can compute the survival signature of a system consisting of two subsystems in either series or parallel configuration, if the survival signatures of those subsystems are available [10]. A generalization of this combinatorial result has also been presented for multi-state systems [23].

The main reason for the introduction of the survival signature is to enable quantification of system reliability, and related statistical inferences, for large real-world systems and networks, for which one normally would not have the full structure function available. We can think here about, for example, about large industrial systems or transportation networks with thousands of components. For such cases, one may need to approximate the survival signature. To do so, it is particularly useful that the survival signature of a coherent system is an increasing function. Approximating the survival signature has received much attention. For example, Behrensdorf et al [4] use percolation theory to exclude areas of the input space of the survival signature where its value does not increase, followed by approximation of the survival signature in the other parts of the input space by

Monte Carlo (MC) methods. They illustrate their method on a model of the Great Britain (GB) electricity transmission network, consisting of 29 nodes of two types, and on a model of the Berlin metro network, consisting of 306 nodes and 350 edges, with the nodes divided into two types based on their degree. Also using MC, Di Maio et al [13] use entropy to direct the sampling towards non-trivial areas of the input space, and they illustrate their method on the same GB electricity transmission network. Recently, Lopes da Silva and Sullivan [20] have presented a powerful method to approximate the survival signature for two-terminal networks with two types of components. They show that each MC replication to estimate the survival signature entails solving a multi-objective maximum capacity path problem, and adapt a Dijkstra-like bi-objective shortest path algorithm to solve this problem. They show the efficiency of their algorithm compared to other approaches, which increases with the size of the network, by application to several networks including a power system, which has 4,000 nodes and 29,336 arcs and includes cycles and self-loops.

Once the survival signature of a system or network has been derived, or approximated, it is a useful tool for a range of objectives. For example, it enables very efficient simulation to learn the system survival function, as presented by Patelli et al [22] and extended by George-Williams et al [17] for inclusion of dependent failures. It is also useful for statistical inference for the system reliability, as learning from data, possibly in combination with the use of expert judgements, is crucial in many applications. If one has data available on the individual component types, then inference on the system's failure time is quite straightforward. Nonparametric Predictive Inference [9], a frequentist approach using few modelling assumptions made possible by the use of imprecise probabilities [3], can be used to derive bounds for the system survival function [10]. The application of Bayesian methods has been presented as well [2], this is particularly useful if one has relatively little data on component failures and therefore wishes to include expert judgements. Walter et al [29] generalized the Bayesian approach combined with the survival signature by using sets of priors, as typically done in theory of robust Bayesian methods. They showed that, by choosing the sets of priors in a specific way, one can enable detection of conflict between prior judgements and data, when data become available and are used to update the prior distributions. This can be of great practical importance, as it can point to prior judgements being too optimistic, hence the system reliability may be substantially lower than was originally thought.

3 Recent developments and challenges

Since its introduction by Coolen and Coolen-Maturi [6], there has been substantial research contributing to the further theory and applicability of survival signature methods. Recently, Coolen-Maturi et al [11] generalised the concept of the survival signature for multiple systems with multiple types of components and with some components shared between systems. A particularly important feature is that the functioning of these systems can be considered at different times, enabling computation of relevant conditional probabilities with regard to a system's functioning conditional on the status of another system with which it shares components. This theory can also be applied to a system which performs multiple functions, which is very important in practice. This has led to a substantial area of research, typically considering specific reliability scenarios or restricted system structures, e.g. Yi et al [30]) consider systems with a monotone structure function.

Further examples of powerful methodology for system reliability quantification enabled by the use of survival signatures include the modelling of dependence between components of different types [14, 17], reliability-redundancy allocation [18], phased-missions [19], component reliability importance measures [16], resilience achieved by swapping components within a system [21], stochastic comparison of different systems [28] and stochastic processes to describe the system reliability over time with varying assumptions on loads or failure processes [5]. A main challenge for applications is the required generalization or adaptation of the survival signature concept for specific scenarios and objectives, ensuring a fruitful field for further research leading to breakthroughs in practical applications, particularly due to the large increase of the sizes of systems and networks for which the reliability can be quantified by the use of the survival signature.

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