# Pricing exotic options in the incomplete market: an imprecise probability method

Ting He*, Frank P.A. Coolen, Tahani Coolen-Maturi

*School of Finance, Capital University of Economics and Business, Beijing, China*

*Department of Mathematical Sciences, Durham University, Durham, United Kingdom*

**Abstract**

This paper considers a novel exotic option pricing method for incomplete markets. Nonparametric Predictive Inference (NPI) is applied to the option pricing procedure based on the binomial tree model allowing the method to evaluate exotic options with limited information and few assumptions. As the implementation of the NPI method is greatly simplified by the monotonicity of the option payoff in the tree, we categorize exotic options by their payoff monotonicity and study a typical type of exotic option in each category, the barrier option and the look-back option. By comparison with the classic binomial tree model, we investigate the performance of our method either with different moneyness or varying maturity. All outcomes show that our model offers a feasible approach to price the exotic options with limited information, which makes it can be utilized for both complete and incomplete markets.

*Keywords:* Imprecise probability; Exotic option; Incomplete market; Nonparametric Predictive Inference; Uncertainty

*Corresponding author

*Email addresses:* ting.he@cueb.edu.cn (Ting He ), frank.coolen@durham.ac.uk (Frank P.A. Coolen), tahani.maturi@durham.ac.uk (Tahani Coolen-Maturi)

*Preprint submitted to APPLIED STOCHASTIC MODELS IN BUSINESS AND INDUSTRY January 11, 2022*

## 1. Introduction

The term 'Exotic option' was used by Rubinstein in 1990 [20], which is a long time after the actual product was presented. Distinguishing from European option and American option collectively known as vanilla option, the exotic option has flexible and complex trading features to meet the particular demands of clients. Financial engineers add additional exercise conditions to the vanilla options to make it exotic. As a derivative financial product, more and more exotic options are produced by financial engineers, like the digital option, the barrier option and the look-back option. However, since the problem of liquidity and asymmetric information exists in the exotic option market, pricing exotic options in the incomplete market becomes a new academic focus, where 'incomplete market' means that there is insufficient information available for investors to determine precise parameter values for the pricing model. In this paper, Nonparametric Predictive Inference (NPI) is implemented in the binomial tree model aiming to price exotic options in incomplete markets.

Nonparametric Predictive Inference (NPI) is an imprecise frequentist statistical framework based on few assumptions, which does not assume prior information, and all inferences are based on the historical data with updating. NPI is an inferential frame work based on the $A_{(n)}$ assumption presented by Hill [14], which has strong consistency properties in the theory of frequentist statistics [3]. Coolen presented the NPI method for Bernoulli random quantities to calculate the upper and lower probabilities [7], and this has been applied to the vanilla option pricing procedure combined with the binomial tree model [12][13]. The development of the NPI methods for vanilla options offers the maximum buying price and the minimum selling price as the bounds of the predictive price results, based only on historical data. To set up the binomial tree, we follow the same assumptions about underlying asset price movements in the model preposed by Cox, Ross and Rubinsein (CRR) [9], by supposing the prices of the underlying asset are a sequence of random quantities, with the movement factors $u$ for upward movement and $d$ for downward movement. Other than using a precise probability for upward price movement, in our model, we inference imprecise probability bounds by assuming that there are $n$ historical underlying asset prices available and $s$ of them increased. By assuming that the underlying asset price follows the binomial tree, the NPI method provides upper and lower probabilities for each time step movement. Suppose that $S_1$ is the underlying asset price at the next future step, and $S_1 = 1$ means that the underlying asset increases at the first future step. Then its corresponding upper and lower probabilities for this event are,

$$\overline{P}\left(S_1 = 1|(n,s)\right) = \frac{s+1}{n+1} \tag{1}$$

$$\underline{P}\left(S_1 = 1|(n,s)\right) = \frac{s}{n+1} \tag{2}$$

One of the advantageous properties of the NPI method is that it predicts the first future step relying on the historical data and adds this predicted value to the data base for next steps prediction. Therefore, the imprecise probabilities are updated along with the prediction procedure. The general formulae for any one step in the binomial tree are,

$$\overline{P}(S_{t+1} = 1|(n+t, s+t-i+1)) = \frac{s+t-i+2}{n+t+1} \tag{3}$$

$$\underline{P}(S_{t+1} = 1|(n+t, s+t-i+1)) = \frac{s+t-i+1}{n+t+1} \tag{4}$$

where $t = 0, \ldots, T$ is the time step during the option validation period, and $i = 1, \ldots, t+1$ is the number of the nodes from the top to the bottom of the binomial tree at time $t$. The payoff function of the vanilla option is denoted as the $g(S_t)$, where $S_t$ is the underlying asset price. The NPI method provides the following lower and upper expectations for the option payoffs,

$$\underline{E}(g(S_t)) = \inf_{p\in\mathcal{P}} E^p(g(S_t))) \tag{5}$$

$$\overline{E}(g(S_t) = \sup_{p\in\mathcal{P}} E^p(g(S_t))) \tag{6}$$

where $\mathcal{P}$ is a set of classical, precise probability distributions corresponding to the NPI approach for Bernoulli data [7], and $E^p$ is the expected value corresponding to a specific precise probability distribution $p \in \mathcal{P}$. By applying this method to to price exotic options, the investor is offered a price interval, of which the maximum buying price is its lower bound and the minimum selling price is its upper bound:

$$\text{call option:} \begin{cases} \underline{E_c} = \text{Payoff} \times \underline{P}(S_t - K), & \text{maximum buying price} \\ \overline{E_c} = \text{Payoff} \times \overline{P}(S_t - K), & \text{minimum selling price} \end{cases}$$

$$\text{put option:} \begin{cases} \underline{E_p} = \text{Payoff} \times \overline{P}(K - S_t), & \text{maximum buying price} \\ \overline{E_c} = \text{Payoff} \times \underline{P}(K - S_t), & \text{minimum selling price} \end{cases}$$

where $S_t$ is the underlying asset price at the exercise time, and $K$ is the strike price, and $\underline{P}$ and $\overline{P}$ are lower and upper probabilities for the event of interest inferred by the NPI method.

In a previous paper, we have derived the closed formulae for European options pricing procedure and found that the new method leads to profit when the real market does not comfort with CRR assumptions that the precise probability can not be inferred from the real market [12]. The minimum selling price for European call options $\overline{V_c}$ is

$$\overline{V_c} = B(0,m)\binom{n+m}{m}^{-1} \sum_{k=\lceil k_c^* \rceil}^{m} [u^k d^{m-k} S_0 - K_c] \binom{s+k}{k}\binom{n-s+m-k-1}{m-k} \tag{7}$$

and the maximum buying price for European call option $\underline{V_c}$ is

$$\underline{V_c} = B(0,m)\binom{n+m}{m}^{-1} \sum_{k=\lceil k_c^* \rceil}^{m} [u^k d^{m-k} S_0 - K_c] \binom{s+k-1}{k}\binom{n-s+m-k}{m-k} \tag{8}$$

The minimum selling price for European put options $\overline{V_p}$ is

$$\overline{V_p} = B(0,m)\binom{n+m}{m}^{-1}\sum_{k=0}^{\lfloor k_p^* \rfloor}[K_p - u^k d^{m-k} S_0]\binom{s+k-1}{k}\binom{n-s+m-k}{m-k} \tag{9}$$

and the maximum buying price for European put option $\underline{V_p}$ is

$$\underline{V_p} = B(0,m)\binom{n+m}{m}^{-1}\sum_{k=0}^{\lfloor k_p^* \rfloor}[K_p - u^k d^{m-k} S_0]\binom{s+k}{k}\binom{n-s+m-k-1}{m-k} \tag{10}$$

where $B(0,m)$ is discount factor for $m$ future steps, $K$ is the strike price of the European option, $u$ and $d$ are the upward and downward movement factors, $\lceil a \rceil$ denotes the maximum integer greater than or equal to $a$, and $\lfloor a \rfloor$ denotes the minimum integer that is less than or equal to $a$. The fundamental idea of the NPI method for European options is to price the minimum selling price by assigning the greatest probability to the movement path with the highest payoff and the second greatest probability to the remaining highest payoff and so on. While the NPI method determines the maximum buying price by assigning the greatest probability to the path with the lowest payoffs and the second greatest probability to the second lowest payoff in the tree .

For American options, there is no closed formula to price the option like the classic CRR model, but we have presented the mathematical description of the American option backward optimization method on the basis of the NPI method [13]. At each time step, the option price $V_t^i$, where $t$ denotes the time step and $i$ denotes the ordered node at time $t$, is derived by comparing the instant value and the discount exception from the next time step.The NPI method for pricing American call options leads to maximum buying price

$$\begin{aligned}
\underline{V_t^i}_{\{i=1\ldots t+1,\ t=0\ldots T-1\}} &= \max\left\{S_t^i - K_c, (1+r)^{-1}\left[\underline{P_t^i}\,\underline{V_{t+1}^i} + (1-\underline{P_t^i})\underline{V_{t+1}^{i+1}})\right]\right\} \\
&= \max\left\{S_t^i - K_c, (1+r)^{-1}\left[\frac{s+t-i+1}{n+t+1}\underline{V_{t+1}^i} + \frac{n-s+i}{n+t+1}\underline{V_{t+1}^{i+1}}\right]\right\} \\
\underline{V_T^i}_{\{i=1\ldots T+1\}} &= \max\{0, S_T^i - K_c\}
\end{aligned} \tag{11}$$

and minimum selling price

$$\begin{aligned}
\overline{V_t^i}_{\{i=1\ldots t+1,\ t=0\ldots T-1\}} &= \max\left\{S_t^i - K_c, (1+r)^{-1}\left[\overline{P_t^i}\overline{V_{t+1}^i} + (1-\overline{P_t^i})\overline{V_{t+1}^{i+1}}\right]\right\} \\
&= \max\left\{S_t^i - K_c, (1+r)^{-1}\left[\frac{s+t-i+2}{n+t+1}\overline{V_{t+1}^i} + \frac{n-s+i-1}{n+t+1}\overline{V_{t+1}^{i+1}}\right]\right\} \\
\overline{V_T^i}_{\{i=1\ldots T+1\}} &= \max\{0, S_T^i - K_c\}
\end{aligned} \tag{12}$$

For pricing American put options, the NPI method leads to maximum buying price derived by the following recursive relation

$$\begin{aligned}
\underline{V_t^i}_{\{i=1\ldots t+1\}} &= \max\left\{K_p - S_t^i, (1+r)^{-1}\left[\overline{P_t^i}\underline{V_{t+1}^i} + (1-\overline{P_t^i})\underline{V_{t+1}^{i+1}}\right]\right\} \\
&= \max\left\{K_p - S_t^i, (1+r)^{-1}\left[\frac{s+t-i+2}{n+t+1}\underline{V_{t+1}^i} + \frac{n-s+i-1}{n+t+1}\underline{V_{t+1}^{i+1}}\right]\right\} \\
\underline{V_T^i}_{\{i=1\ldots T+1\}} &= \max\{0, K_p - S_T^i\}
\end{aligned} \tag{13}$$

and minimum selling price

$$\begin{aligned}
\overline{V_t^i}_{\{i=1\ldots t+1\}} &= \max\left\{K_p - S_t^i, (1+r)^{-1}\left[\underline{P_t^i}\overline{V_{t+1}^i} + (1-\underline{P_t^i})\overline{V_{t+1}^{i+1}}\right]\right\} \\
&= \max\left\{K_p - S_t^i, (1+r)^{-1}\left[\frac{s+t-i+1}{n+t+1}\overline{V_{t+1}^i} + \frac{n-s+i}{n+t+1}\overline{V_{t+1}^{i+1}}\right]\right\} \\
\overline{V_T^i}_{\{i=1\ldots T+1\}} &= \max\{0, K_p - S_T^i\}
\end{aligned} \tag{14}$$

where $t = 0, \ldots, T$ is the time step during the option validation period, and $i = 1, \ldots, t+1$, is the number of the nodes from the top to the bottom of the binomial tree at time $t$. Then $\underline{V_t^i}$ and $\overline{V_t^i}$ are the lower and upper predicted value of American option for node $i$ at time $t$ $r$ is the discount rate.

The NPI method has been utilized to price these vanilla options in incomplete markets, and showed good performance [12]; [13]. There is a common characteristic of all vanilla options, which is a monotonic payoff in the binomial tree. It means that the payoffs of the European and the American options are monotone functions of the number of upward movements. However, the particular payoff definition of some types of exotic options jeopardize the consistency of payoff monotonicity. In terms of the payoff monotonicity of the exotic options, we can categorize the exotic options in two types, ones with monotonic payoffs like the barrier option and ones with non-monotonic payoffs like the look-back option. Therefore, in this paper we will present the idea of pricing a variety of exotic options according to their payoff monotonicities. The payoff monotonicity is discussed in the Section 2, as the payoff for some kinds of exotic options can be non-monotone while for the vanilla options this kind of problem does not exist. In Section 3, a type of exotic option with monotonic payoff, the barrier option, is priced by the NPI method, and the performance of the new method is studied by simulation. In Section 4, we investigate the NPI method for the look back option with the floating strike price, which is a typical type of exotic option with non-monotonic payoff. The conclusions and remarks are discussed in Section 5.

## 2. Payoff monotonicity

So far, we have applied the NPI method to the European and American options. Since the payoffs of the European and the American options are monotone functions of the number of upward movements, it is less complicated to determine the upper and lower probabilities. The payoff of the European call option is $[S_T - K_c]^+$, and the payoff of the European put option is $[K_p - S_T]^+$. As in the binomial tree, the top node at time $T$ has the largest value of $S_T$, $[S_T - K_c]^+$ has the largest value at the top node and decreases as the node moves to the bottom of the tree. While $[K_p - S_T]^+$ has the lowest value at the top node at time $T$ and increases as the node moves to the bottom of the tree. For the American options, although for each path in the binomial tree, the exercise time $\tau$ can be different, the payoffs at $\tau$ are still monotonic. Based on the definition of the American options, $[S_\tau - K_c]^+$ is the payoff for the American call option, and $[K_p - S_\tau]^+$ is the payoff for the American put option. As $\tau$ is the best time to exercise the American option to get the optimal payoff for each path in the binomial tree, $[S_\tau - K_c]^+$ has the largest value at the top node at time $\tau$ and decreases as the node move to the bottom of the binomial tree. $[K_p - S_\tau]^+$ has the lowest value at the top node at time $\tau$ and increases as the node moves to the bottom of the tree. So the payoff of the American option is also monotonic.

Applying the NPI method to an option with monotonic payoffs is less complicated than to an option with non-monotonic payoffs. For instance, when we want to calculate the upper expected payoff of a call option with monotonic payoffs, we can assign the upper probability from Equation (3) to each one-time-step path of the upward movement in the binomial tree to get the result. As in a one time step tree, the probability for upward movement and the probability for downward movement are summed to one, when we assign the upper probability to upward movement path, the lower conjugacy probability is assigned to downward movement path. Correspondingly, if the lower expected payoff is needed, we can compute it by assigning the lower probability from Equation (4) to each one-time-step path of the upward movement in the binomial tree. However, if the payoffs of an option are not monotonic, the upper and lower expected payoffs cannot be calculated by assigning the upper and lower probabilities, which need to be determined by a more detailed search over the set of probabilities $\mathcal{P}$. This does not mean that we cannot use the NPI method to derive the lower and upper expected value of non-monotonic payoffs. In the following sections, we will illustrate the NPI method for exotic options in two categories, with and without monotonic payoffs, and in each category one typical exotic option is studied, the barrier option and the look-back option.

## 3. Exotic option with monotonic payoffs: the barrier option

The barrier option is a well-known exotic option with monotonic payoffs. This kind of option has a unique feature distinguishing it from the vanilla option, namely that a barrier for the underlying asset price is predetermined. This barrier for the asset price justifies the option's validation that if the future asset price reaches the barrier, either this option expires or be valid immediately. Merton [17] first presented the down and out option in 1973. Now there are two classes of the barrier option, "knock-in" and "knock-out" barrier options. The "knock-in" option has a barrier making the option exercisable, while the barrier of the "knock-out" option leads to the expiration of the option. According to the initial underlying asset price, both "in" and "out" options are separated into "up" and "down" options. In total, there are eight types of barrier options. Many scholars present a variety of methods to price the barrier option. Cox and Rubinstein [8] illustrated this type of barrier option pricing model based on the CRR model in 1985. Rubinstein and Reiner [21] listed formulae for the eight different barrier options in a continuous time model. Boyle and Lau [5] used the binomial lattices to price the barrier option and found its convergence of prices of barrier options. In 1996, Reimer and Sandmann [19] explained the formulae for all types of barrier options including European style and the American style, which are all set up in the risk-neutral world. In 2006, a modified standard binomial method which can price the American type barrier option was introduced by Gaudenzi and Lepellere [10], which is more efficient than the CRR model and can be used in the trinomial tree method as well. Appolloni et al. [2] explored the binomial lattice method to evaluate the step double barrier options.

We denote the barrier of asset price as $S_b$, for the knock-in options, the options are valid when the stock price is less than $S_b$ for the down-and-in option or greater than $S_b$ for the up-and-in option. Here we use the indicator $\mathbb{1}$ to describe the barrier, so for the down-and-out option, the barrier is denoted as $\mathbb{1}_{\{S_t > S_b, t \in (0,\dots,T)\}}$ and for the up-and-in option, the barrier is denoted as $1 - \mathbb{1}_{\{S_t < S_b, t \in (0,\dots,T)\}}$. According to the payoffs for the call and put options, we can define the knock-in options as follows.

Knock-in options

$$\text{down-and-in} \begin{cases} [S_T - K_c]^+ \left(1 - \mathbb{1}_{\{S_t > S_b, t \in (0,\dots,T)\}}\right), & \text{Call} \\ [K_p - S_T]^+ \left(1 - \mathbb{1}_{\{S_t > S_b, t \in (0,\dots,T)\}}\right), & \text{Put} \end{cases}$$

For a down-and-in option, as long as the stock price during the option valid period

$S_t$ goes down and reaches the barrier value $S_b$, the option holder can get the payoff as $[S_T - K_c]^+$ for the call option or $[K_p - S_T]^+$ for the put option at maturity. The definition of the up and in option is

$$\text{up-and-in} \begin{cases} [S_T - K_c]^+ \left(1 - \mathbb{1}_{\{S_t < S_b, t \in (0,\ldots,T)\}}\right), & \text{Call} \\ [K_p - S_T]^+ \left(1 - \mathbb{1}_{\{S_t < S_b, t \in (0,\ldots,T)\}}\right), & \text{Put} \end{cases}$$

For an up-and-in option, as long as the stock price during the option valid time $S_t$ goes up and reaches the barrier value $S_b$, the corresponding option is immediately valid and offers the payoff $[S_T - K_c]^+$ for the call option or $[K_p - S_T]^+$ for the put option at maturity.

For the knock-out options, the option is expired once the stock price $S_t$ touches the barrier $S_b$. Thus, the down-and-out option is valid when $\mathbb{1}_{\{S_t > S_b, t \in (0,\ldots,T)\}}$, and the up-and-out option is valid when $\mathbb{1}_{\{S_t < S_b, t \in (0,\ldots,T)\}}$. Due to the barrier, the formulae of the knock-out options are given below.

Knock-out options

$$\text{down-and-out} \begin{cases} [S_T - K_c]^+ \mathbb{1}_{\{S_t > S_b, t \in (0,\ldots,T)\}}, & \text{Call} \\ [K_p - S_T]^+ \mathbb{1}_{\{S_t > S_b, t \in (0,\ldots,T)\}}, & \text{Put} \end{cases}$$

For a down-and-out option, if the stock price during the option validation $S_t$ is always greater than the barrier value $S_b$, then the option holder can get the payoff as $[S_T - K_c]^+$ for the call option or $[K_p - S_T]^+$ for the put option in the end.

$$\text{up-and-out} \begin{cases} [S_T - K_c]^+ \mathbb{1}_{\{S_t < S_b, t \in (0,\ldots,T)\}}, & \text{Call} \\ [K_p - S_T]^+ \mathbb{1}_{\{S_t < S_b, t \in (0,\ldots,T)\}}, & \text{Put} \end{cases}$$

When it comes to an up-and-out option, during the option validation, as long as the stock price $S_t$ always holds a lower value than the barrier value $S_b$, the option holder can the payoff as $[S_T - K_c]^+$ for the call option or $[K_p - S_T]^+$ for the put option at maturity.

From the definition of the barrier option, we can tell that to evaluate a barrier option

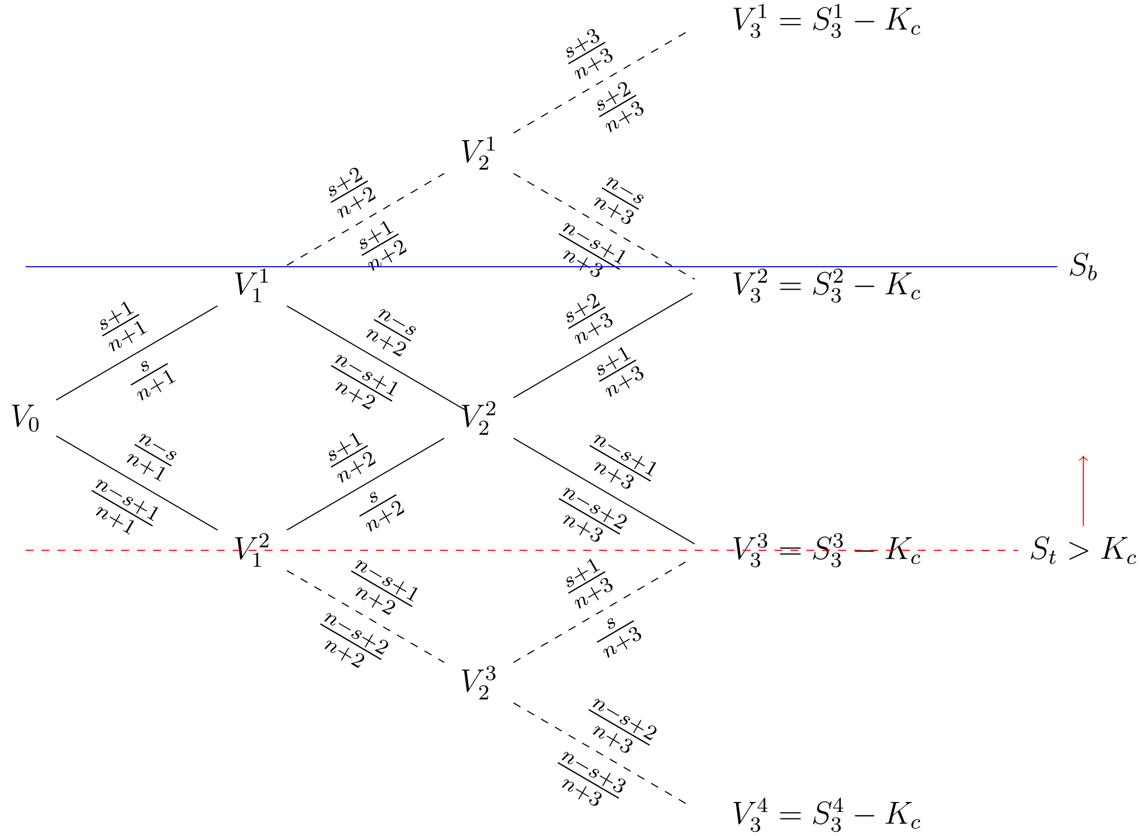


Figure 1: The binomial tree based on the NPI method for an up-and-out call option

we need to monitor the underlying asset regularly during the option life period, and as long as the option reaches the bound the option is either valid or expired. The NPI method can also be applied to this option to price the barrier option even if there is limited information available in the market. For the knock-out type of option, even though there is no closed-form formula, we can use the backward valuation method to get the expected option price.

Figure 1 displays a up-and-out call option. The payoffs are monotonic with the path, and the probabilities of the NPI boundary prices of the barrier option for each path are the same for the corresponding vanilla options. However, due to the bound $S_b$, the path included in the pricing procedure is reduced, which means that the paths having the asset price greater or equal to the $S_b$ are excluded, even though they hold a positive payoff. Referring to Figure 1, only the paths drawn in solid line for all time steps are involved in the pricing evaluations.

The details of evaluating this type of exotic option are based on the backward optimization method. We start from the maturity payoff $[S_T - K_c]^+$ for the call option and $[K_p - S_T]^+$ for the put option, rolling back to the initial time. At each time step, we check the condition of the "knock-in" or "knock-out" option. For example, if there is an up-and-out $m$ period call option with the barrier $S_b$, then we can get the maturity payoff at each node $i$ as $[S_T^i - K_c]^+$ with $i \in \{1, \dots, T+1\}$ and $T = m$, and we check if the underlying asset price at maturity $S_T^i$ follows the condition $S_T^i < S_b$. If not, then the option value at that node is immediate equal to zero. Thus, the payoff of the whole tree is $V_T^i = [S_T^i - K_c]^+ \mathbb{1}_{\{S_T < S_b\}}$. Then we move backward to the time step before the maturity $T-1$. At $T-1$ the option value is the expectation at maturity after the discount procedure if the spot price is less than $S_b$. Otherwise, the option value equals to zero, thus, $V_{T-1}^i = B(T-1,T)[S_T^i - K_c]^+ \mathbb{1}_{\{S_T < S_b\}} \mathbb{1}_{\{S_{T-1} < S_b\}}$. According to the NPI method, we can get the upper and lower expectations based on $n$ historical stock price data with $s$ increased prices which is the same procedure as we applied for American option pricing [13]. By assigning the lower probability formulated, given in Equation (4), to the binomial tree at maturity, we obtain the lower expected value at maturity $\underline{E}[S_T^i - K_c]^+ = \underline{P_{T-1}^i}[S_T^i - K_c]^+ + (1 - \underline{P_t^i})[S_T^{i+1} - K_c]^+)$, and the upper expected value can be obtained by assigning the upper probability formulated as Equation (3). Then the lower and upper expected values lead to two boundaries of the option value at time $T-1$ by discounting. After applying the same procedure at every time step, we can get two initial boundary option values, which we consider to be the maximum buying price and the minimum selling price. The investor would like to sell the option if the quote price is greater than the minimum selling and buy the option if the quote price is less than the maximum buying price. If the quote price is in the interval, then it is reasonable meaning that no trade in appealing. For an up-and-out call option, the NPI method leads to maximum buying price derived by the following recursive relation

$$\begin{aligned}
\underline{V_t^i}_{\{t\in\{0\ldots m-1\}\ i\in\{1\ldots t+1\}\}} &= B(t,t+1)\left[\underline{P_t^i}\ \underline{V_{t+1}^i} + (1-\underline{P_t^i})\underline{V_{t+1}^{i+1}}\right]\mathbb{1}_{\{S_t^i<S_b\}} \\
&= (1+r)^{-1}\left[\frac{s+t-i+1}{n+t+1}\underline{V_{t+1}^i} + \frac{n-s+i}{n+t+1}\underline{V_{t+1}^{i+1}}\right]\mathbb{1}_{\{S_t^i<S_b\}} \qquad (15) \\
\underline{V_T^i}_{\{T=m\ i\in\{1\ldots T+1\}\}} &= [S_T^i - K_c]^+\mathbb{1}_{\{S_T^i<S_b\}}
\end{aligned}$$

and minimum selling price

$$\begin{aligned}
\overline{V_t^i}_{\{t\in\{0\ldots m-1\}\ i\in\{1\ldots t+1\}\}} &= B(t,t+1)\left[\overline{P_t^i}\ \overline{V_{t+1}^i} + (1-\overline{P_t^i})\overline{V_{t+1}^{i+1}}\right]\mathbb{1}_{\{S_t^i<S_b\}} \\
&= (1+r)^{-1}\left[\frac{s+t-i+2}{n+t+1}\overline{V_{t+1}^i} + \frac{n-s+i-1}{n+t+1}\overline{V_{t+1}^{i+1}}\right]\mathbb{1}_{\{S_t^i<S_b\}} \\
\overline{V_T^i}_{\{T=m\ i\in\{1\ldots T+1\}\}} &= [S_T^i - K_c]^+\mathbb{1}_{\{S_T^i<S_b\}} \qquad (16)
\end{aligned}$$

For an up-and-out put option, we assignment the imprecise probability bounds but the other way around to that for the up-and-out call option to calculate the lower and upper expected prices, which the maximum buying price is obtained by assigning the upper probability, and the corresponding minimum selling price is obtained by assigning the lower probability in the binomial tree. Same ass for the call option presented above, there are no closed-form formulae for the put option. The NPI method leads to maximum buying price for the put option

$$\begin{aligned}
\underline{V_t^i}_{\{t\in\{0\ldots m-1\}\ i\in\{1\ldots t+1\}\}} &= B(t,t+1)\left[\overline{P_t^i}\ \underline{V_{t+1}^i} + (1-\overline{P_t^i})\underline{V_{t+1}^{i+1}}\right]\mathbb{1}_{\{S_t^i<S_b\}} \\
&= (1+r)^{-1}\left[\frac{s+t-i+2}{n+t+1}\underline{V_{t+1}^i} + \frac{n-s+i-1}{n+t+1}\underline{V_{t+1}^{i+1}}\right]\mathbb{1}_{\{S_t^i<S_b\}} \\
\underline{V_T^i}_{\{T=m\ i\in\{1\ldots T+1\}\}} &= [K_p - S_T^i]^+\mathbb{1}_{\{S_T^i<S_b\}} \qquad (17)
\end{aligned}$$

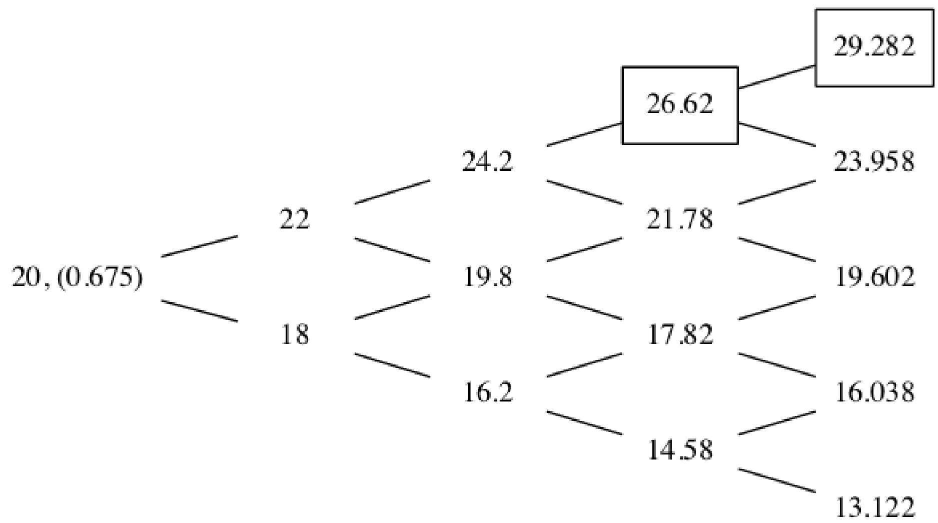


Figure 2: The binomial tree of an up-and-out call option

and minimum selling price

$$
\begin{aligned}
\overline{V^i_t}_{\{t\in\{0\ldots m-1\}\ i\in\{1\ldots t+1\}\}} &= B(t,t+1)\left[\underline{P^i_t}\,\overline{V^i_{t+1}} + (1-\underline{P^i_t})\overline{V^{i+1}_{t+1}}\right]\mathbb{1}_{\{S^i_t<S_b\}} \\
&= (1+r)^{-1}\left[\frac{s+t-i+1}{n+t+1}\overline{V^i_{t+1}} + \frac{n-s+i}{n+t+1}\overline{V^{i+1}_{t+1}}\right]\mathbb{1}_{\{S^i_t<S_b\}} \qquad (18) \\
\overline{V^i_T}_{\{T=m\ i\in\{1\ldots T+1\}\}} &= [K_p - S^i_T]^{+}\mathbb{1}_{\{S^i_T<S_b\}}
\end{aligned}
$$

When it comes to the down-and-out options, only the barrier changes to $S^i_t > S_b$, other than that, the probability assignment and payoff are the same as the up-and-out barrier options.

**Example 3.1**

By using the statistical software R, we determine the NPI maximum buying and minimum selling prices for an up-and-out call option with the strike price $K = 21$ based on $n = 50$ and $s = 30$ historical data. In this example, the underlying asset with an initial price $S_0 = 20$ has a barrier $S_b = 26$. Then any path that reaches the barrier of the asset price is not included in the option evaluation. As the stock price movement is a Bernoulli random quantity, the stock price moves either up with the factor $u = 1.1$ or down with the factor $d = 0.9$, and the asset price at each node in the binomial tree is determined. In

Figure 2, the two nodes higher than the barrier are in the boxes, which are $S_3^1 = 26.62$ and $S_4^1 = 29.282$. So the paths with the nodes in the boxes are not involved in the evaluation holding the value of zero. Then we can get the option value at every node of the binomial tree. Here the discount rate is a constant value equal to $r_f = 0.02$. After doing the backward evaluation until the initial time, we get the expected price of this up-and-out barrier option shown as the value 0.675 in the parenthesis in Figure 2.

**Example 3.2**

To assess the predictive performance of the NPI method, we bring the classical binomial tree model (CRR) presented by Cox, Ross and Rubinstein in 1979 [9] into the comparison. We evaluate the barrier options for the same underlying asset by both the NPI method given $n = 50$ historical data with $s = 25$ increases of the price, and the CRR model assuming $q = 0.5$. This four time-step barrier options are contingent on the underlying asset with the initial price $S_0 = 20$, the upward movement factor $u = 1.1$ and the downward movement factor $d = 0.9$. For a more detailed investigation, the barrier options with different moneyness are priced in this example meaning this assessment covers in-the-money, at-the-money and out-of-the-money options. The assessment result is plotted in Figure 3.

Figure 3 shows that the CRR price is in the interval between the maximum buying and the minimum selling prices from the NPI method for in-the-money, at-the-money as well as out-of-the-money options. However, if the option is deep in-the-money, then the CRR model intends to forecast a higher price than the minimum selling price from the NPI method. This is because when the NPI method calculates the minimum selling price, it assigns the greatest probability to the top node that has no positive payoff because of the barrier. Then the NPI method assigns the second greatest probability to the second node from the top that may have no positive payoff as well. Overall, the NPI method in this calculation assigns more probabilities to all upward movements not only ones have positive value but also ones with zero payoffs due to the barrier. Therefore, the NPI method adds slightly more penalty to the barrier for the up and out call option than the CRR model

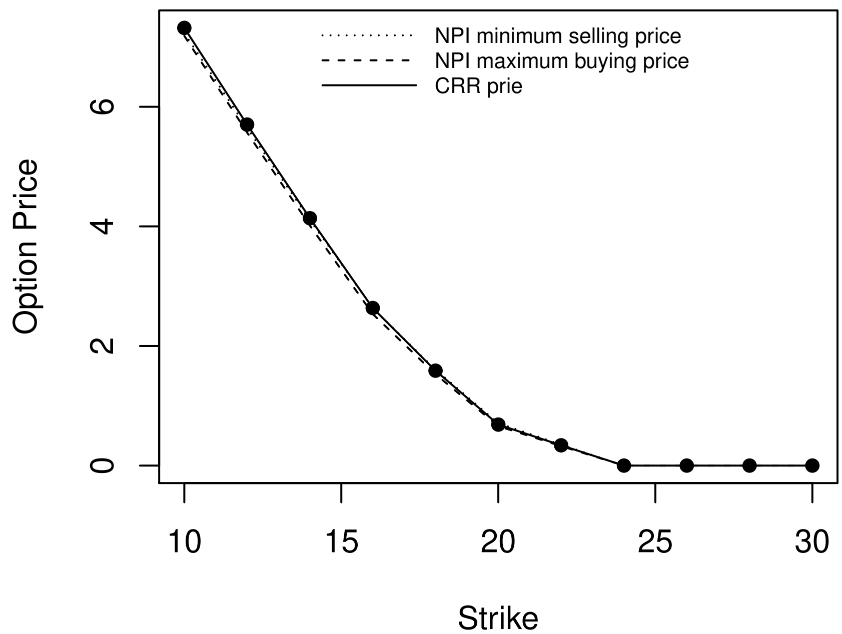


Figure 3: The comparison of the NPI method and the CRR model ($n = 50$)

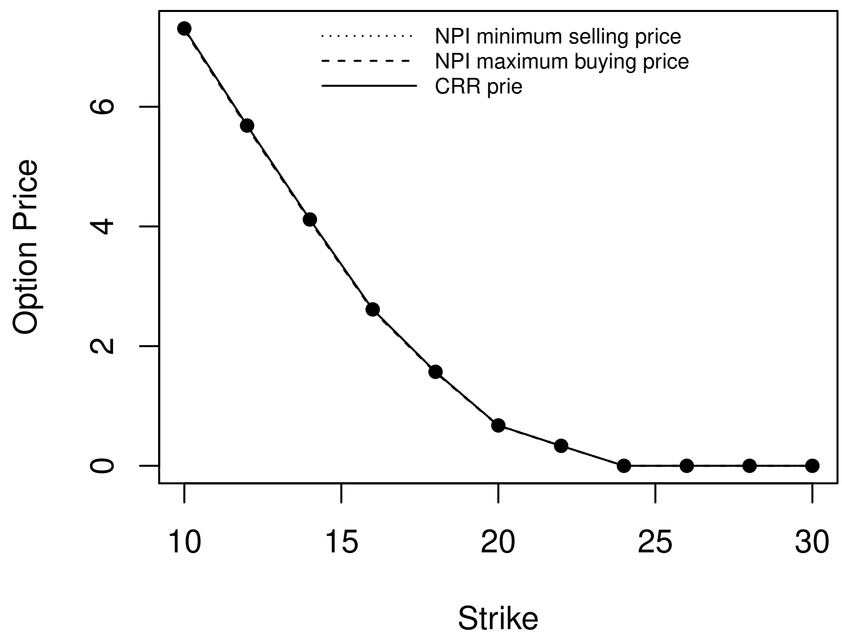


Figure 4: The comparison of the NPI method and the CRR model ($n = 252$)

does, and this leads to the result that the minimum selling price of deep in-the-money option is less than the CRR model. In other words, this shows the NPI method pays more attention to the barrier effect, especially for the deep in-the-money option. This fits the intuitive sense of the investment that the barrier effects more toward the option with positive payoffs than the option with no positive payoff, especially in the incomplete market, since the incompleteness of market enhances the effect of the barrier on the options.

We also find that the sufficient size of historical data makes the NPI result more precise. Figure 4 shows the result based on historical data size $n = 252$ and $s = 126$. From the plot, it is clear that the NPI interval asymptotically approach to a precise value when there is sufficient historical information, and this precise value is identical to the CRR prediction for all kinds of moneyness of options. This result illustrates that the NPI method offers the same prediction for the barrier option as the CRR model does in the complete market.

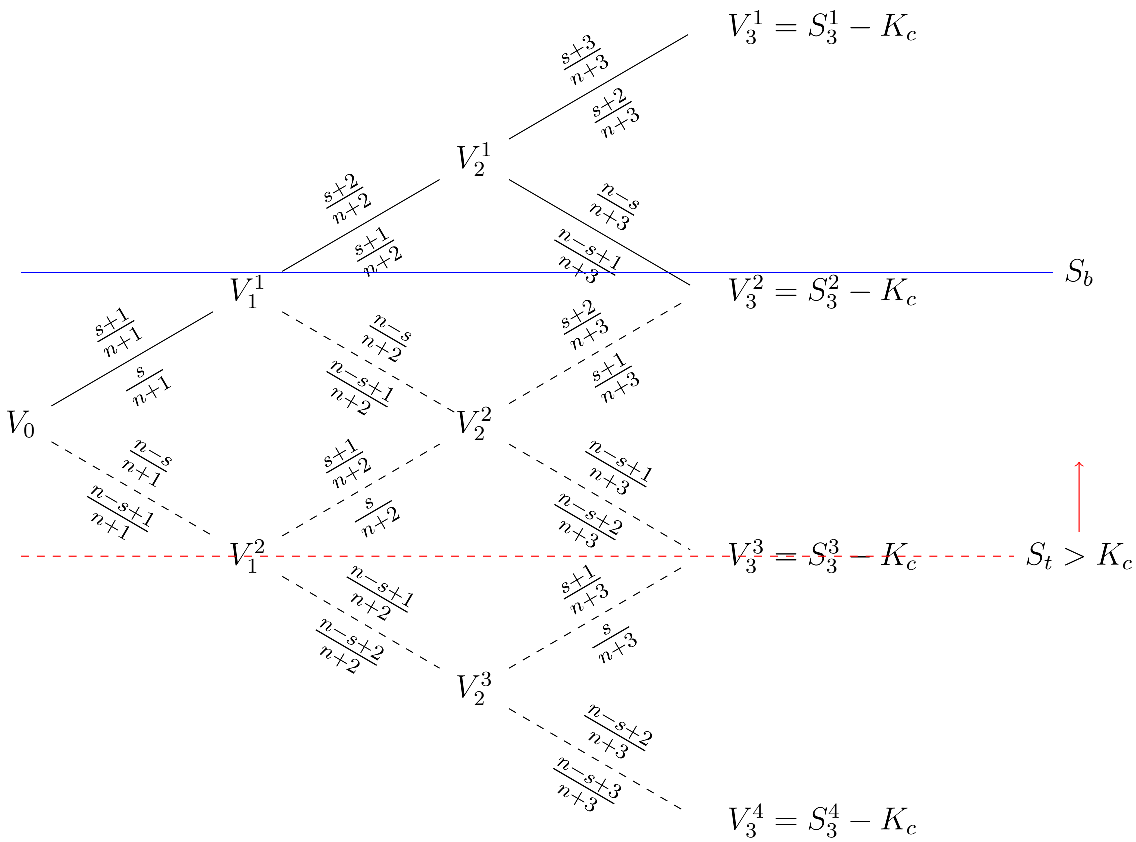


Figure 5: The binomial tree based on the NPI method for an up-and-in call option

Unlike the knock-out option, it is time-consuming to predict the knock-in option only with the backward optimization method. It is easier to illustrate why simply using the backward optimization is not efficient in an example. Figure 5 shows an up-and-in call option with the barrier of the asset price $S_b$. Based on the definition of the up-and-in call option, as $S_2^1$ and $S_3^1$ are higher than the barrier price $S_b$, the only two paths in the evaluation are $V_0 \to V_1^1 \to V_2^1 \to V_3^1$ and $V_0 \to V_1^1 \to V_2^1 \to V_3^2$. Unlike the knock-out barrier option, we need to know the valid path before making the prediction. However, the backward optimization method cannot pick the valid pathes that is across the barrier. In this example, the paths ended up with $V_3^2$ are $V_0 \to V_1^1 \to V_2^1 \to V_3^2$ which is valid, and $V_0 \to V_1^1 \to V_2^2 \to V_3^2$ which is invalid because of barrier. By using only the backward optimization method, the path $V_0 \to V_1^1 \to V_2^2 \to V_3^2$ is still included in the evaluation even though it is invalid, because $V_3^2$ has a positive payoff. This roll-back procedure would not stop until the inital time when it is realized that this path has never come cross the barrier to be validated. Thus, using only the backward optimization method is next to impossible.

To avoid the problem discussed above and simplify the calculation procedure, we put the thought of European option pricing into the evaluation. Assuming at time $t$ the stock price $S_t^i$ is the first node of each path from the initial time qualified with the barrier condition, then we can see option value at this node as a vanilla European option with the same strike price but different maturity $T - t$. After getting all the option values at every first valid node in the tree, we use the backward optimization method to roll back to the initial time and get the expected option price. For instance, for the call option listed in Figure 5, as $V_2^1$ is the first node that greater than $S_b$, we first see it as an one-step European call option with the initial stock price $S_2^1$, then do the backward optimization method to get the predicted option price. Using Equations (8) and (10) for buying position and Equations (7) and (9) for selling position, we can get the option value at node $V_2^1$. One thing we need to pay attention to that for this auxiliary European option, the historical data is $n+t$, and the successful historical data is $s+t-i+1$. Then by applying the backward optimization method we obtain the expected value $V_0$. In summary, it is crucial to find the node of the first time across the barrier $S_t^i$, and the paths subordinate to this node are included in the auxiliary European evaluation, otherwise are valued by the backward optimization method. Generally, this pricing procedure based on the NPI method leads to maximum buying price for the call option.

$$\underline{V_t^i}_{\{t\in\{0\ldots m-1\}\ i\in\{1\ldots t+1\}\}} = \begin{cases} B(t,t+1)\left[\underline{P_t^i}\ \underline{V_{t+1}^i} + (1-\underline{P_t^i})\underline{V_{t+1}^{i+1}}\right] \\ \text{If } S_t^i < S_b \text{ and } S_{t-1}^i < S_b \\ B(t,T)\binom{m+n+t}{m}^{-1}\sum_{k=\lceil k_c^*\rceil}^{m-t}[u^k d^{m-t-k} S_t^i - K_c] \\ \times\binom{s+t-i+k}{k}\binom{n-s-t+i+m-k-1}{m-k} \text{If } S_t^i \geq S_b \text{ and } S_{t-1}^i < S_b \end{cases} \tag{19}$$

$$\underline{V_T^i}_{\{T=m\ i\in\{1\ldots T+1\}\}} = [S_T^i - K_c]^+\left(1 - \mathbb{1}_{\{S_t<S_b, t\in(0,\ldots,T)\}}\right) \tag{20}$$

and minimum selling price

$$\overline{V_t^i}_{\{t\in\{0\ldots m-1\}\ i\in\{1\ldots t+1\}\}} = \begin{cases} B(t,t+1)\left[\overline{P_t^i}\ \overline{V_{t+1}^i} + (1-\overline{P_t^i})\overline{V_{t+1}^{i+1}}\right] \\ \text{If } S_t^i < S_b \text{and } S_{t-1}^i < S_b \\ B(t,T)\binom{m+n+t}{m}^{-1}\sum_{k=\lceil k_c^*\rceil}^{m-t}[u^k d^{m-t-k} S_t^i - K_c] \\ \times\binom{s+t-i+k+1}{k}\binom{n-s-t+i+m-k-2}{m-k} \text{If } S_t^i \geq S_b \text{ and } S_{t-1}^i < S_b \end{cases} \tag{21}$$

$$\overline{V_T^i}_{\{T=m\ i\in\{1\ldots T+1\}\}} = [S_T^i - K_c]^+\left(1 - \mathbb{1}_{\{S_t<S_b, t\in(0,\ldots,T)\}}\right) \tag{22}$$

To price an up-and-in put option, the price procedure for the up-and-in call option can be used by adjusting the payoffs according to the put option payoff definition. The option maturity value equals to $[K_p - S_T^i]^+ \times \left(1 - \mathbb{1}_{\{S_t<S_b, t\in(0,\ldots,T)\}}\right)$. Before the maturity, the option value for each node is either the discount value rolling back from the next time steps option value or equal to the European put option with the maturity $T-t$ calculated based on $n+t$ historical data among them $s+t-i+1$ successful, which depends on the node is whether before the first qualified barrier node or not. The NPI method for the up-

and-in option to the maximum buying price and the minimum selling price are formulated as below.

The maximum buying price for the put option is

$$\underline{V_t^i}_{\{t\in\{0\dots m-1\}\ i\in\{1\dots t+1\}\}} = \begin{cases} B(t,t+1)\left[\overline{P_t^i}\ \underline{V_{t+1}^i} + (1-\overline{P_t^i})\underline{V_{t+1}^{i+1}}\right] \\ \text{If } S_t^i < S_b \text{ and } S_{t-1}^i < S_b \\ B(t,T)\binom{m+n+t}{m}^{-1}\sum_{k=\lceil k_c^* \rceil}^{m-t}[K_p - u^k d^{m-t-k} S_t^i] \\ \times\binom{s+t-i+k+1}{k}\binom{n-s-t+i+m-k-2}{m-k}\text{If } S_t^i \geq S_b \text{ and } S_{t-1}^i < S_b \end{cases} \tag{23}$$

$$\underline{V_T^i}_{\{T=m\ i\in\{1\dots T+1\}\}} = [K_p - S_T^i]^+\left(1 - \mathbb{1}_{\{S_t<S_b, t\in(0,\dots,T)\}}\right) \tag{24}$$

The minimum selling price for the put option is

$$\overline{V_t^i}_{\{t\in\{0\dots m-1\}\ i\in\{1\dots t+1\}\}} = \begin{cases} B(t,t+1)\left[\underline{P_t^i}\ \overline{V_{t+1}^i} + (1-\underline{P_t^i})\overline{V_{t+1}^{i+1}}\right] \\ \text{If } S_t^i < S_b \text{ and } S_{t-1}^i < S_b \\ B(t,T)\binom{m+n+t}{m}^{-1}\sum_{k=\lceil k_c^* \rceil}^{m-t}[K_p - u^k d^{m-t-k} S_t^i] \\ \times\binom{s+t-i+k}{k}\binom{n-s-t+i+m-k-1}{m-k}\text{If } S_t^i \geq S_b \text{ and } S_{t-1}^i < S_b \end{cases} \tag{25}$$

$$\overline{V_T^i}_{\{T=m\ i\in\{1\dots T+1\}\}} = [K_p - S_T^i]^+\left(1 - \mathbb{1}_{\{S_t<S_b, t\in(0,\dots,T)\}}\right) \tag{26}$$

To price the down-an-in barrier option, we change the first valid node of the underlying asset price to the first node that the underlying asset price is lower or equal to the barrier at time $t$ and the indicator function at maturity to $\left(1 - \mathbb{1}_{\{S_t>S_b, t\in(0,\dots,T)\}}\right)$.

**Example 3.3**

Example 3.3 is an up-and-in call option in buying position based on the same underlying asset as that in the pervious examples. The barrier of the underlying asset is $S_b = 23$, so any path contains asset price greater or equal to 23 are included in the pricing procedure. In Figure 6, there is the binomial tree of this option. The nodes in the box are the two cases that the underlying asset price first over the barrier. Let us look at first node $S_2^1 = 24.2$.

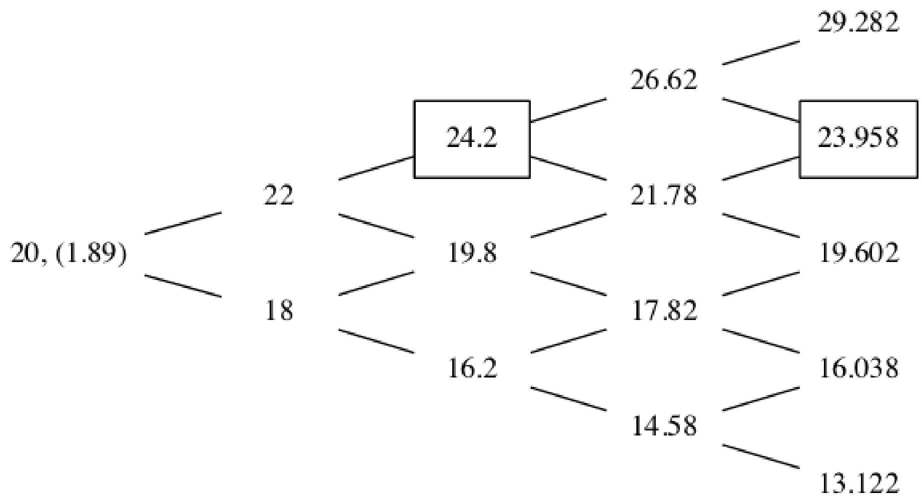


Figure 6: The binomial tree of an up-and-in call option

When the underlying asset price moves to this price, the paths having this node are included in the option price evaluation. So as we described the up-and-in option evaluation, we first compute the option value at this node by seeing it as a vanilla European option with the initial stock price $S_0 = 24.2$ and maturity $m = 2$. Then we straightforwardly roll this option value back to the initial time. The second node in the box is $S_4^2 = 23.958$. This is a maturity node, so we use the backward optimization method to get the initial expected value. However, we would like to highlight one point that as the $S_4^2 = 23.958$ is also included in the paths containing $S_2^1 = 18$, so the two backward procedures of the paths that have $S_4^2$ as the first node over the barrier are $20 \to 22 \to 19.8 \to 21.78 \to 23.958$ and $20 \to 18 \to 19.8 \to 21.78 \to 23.958$. After pricing, the maximum buying price of this up-and-in barrier call option is 1.89 shown in the parenthesis in Figure 6.

**Example 3.4**

To access the performance of the NPI method for the knock in options, we evaluate the up-and-in call option based on the same underlying asset as that in Example 3.3, while the strike price now is a sequence of values to test the method performance according to different moneyness. The first study is on the basis of $n = 50$ historical data and the historical number of upward movements $s = 25$. The outcomes from both the CRR

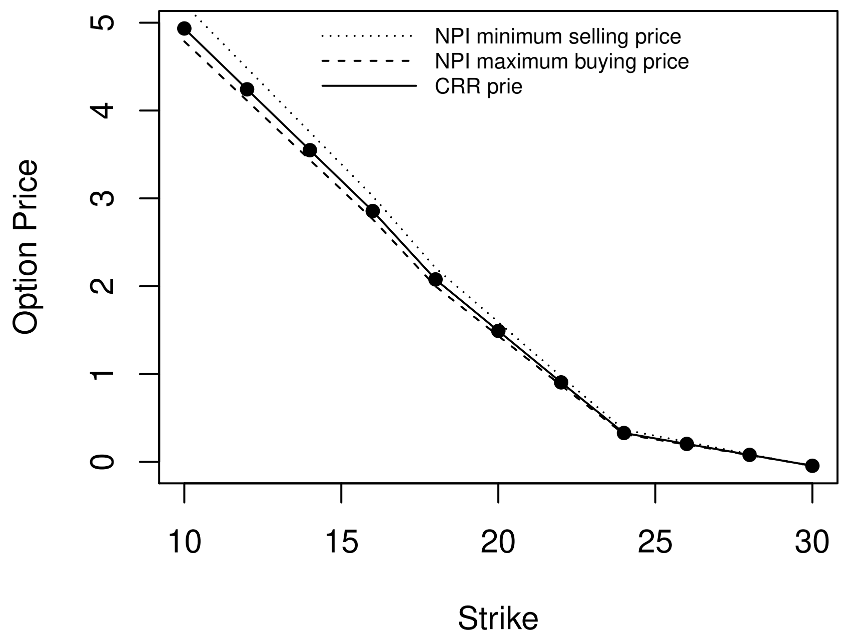


Figure 7: The comparison of the NPI method and the CRR model ($n = 50$)

model and the NPI method are plotted in Figure 7. By comparing the evaluated prices from the CRR model and the NPI method, it is clear when the background information is limited, the interval of the NPI prices is less precise and contains the CRR result. Since this example is based on the up-and-in option, for the deep in the money call option, the barrier effect is stronger than that on the options with other moneyness, because this kind of option holds more paths with positive payoffs than the options with other moneyness. Thus, the barrier will have higher plausibility to jeopardize the validation of the positive payoff paths causing more uncertainty in the evaluation. The NPI method considers this barrier effect problem and reflects it by the precision of the interval while the CRR model does not concern it. The same effect can also happens to the down-and-in put option, and we believe the NPI method will provide a wider price interval for deep in the money options than options with other moneyness.

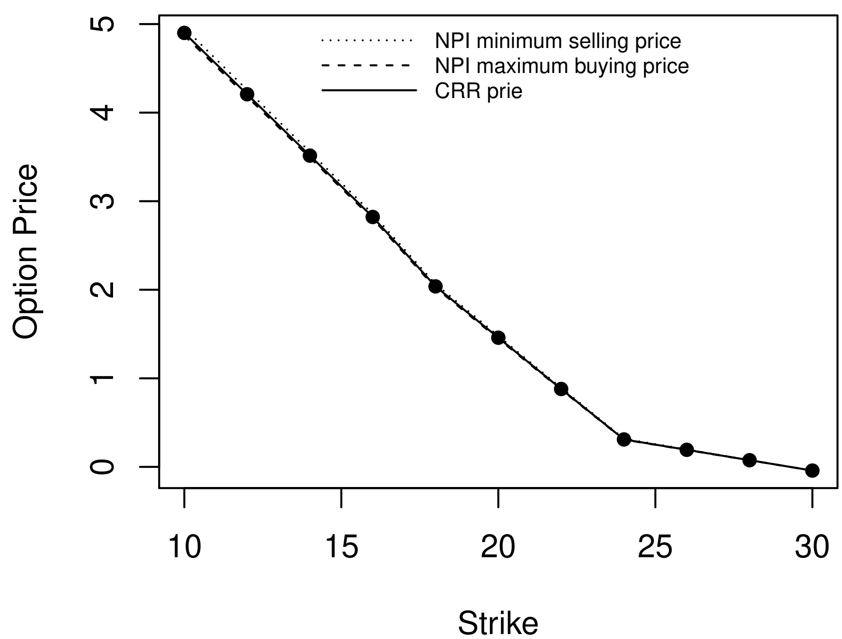


Figure 8: The comparison of the NPI method and the CRR model ($n = 252$)

Another simulation with more sufficient historical information $n = 252$, $s = 126$ is also investigated. Figure 8 indicates that when the information is fully available the NPI method delivers the same explicit result as the CRR does.

Overall, even though the barrier option is a path-dependent exotic option, the NPI method is applied directly to the option evaluation based on the binomial tree model because of the payoff monotonicity. Compared with the CRR model, the NPI method does the prediction based on the available historical data with few assumptions, which is superior to the CRR model when the information is limit. Especially for the incomplete market, when the information is insufficient, the NPI method not only offers the interval results considering the risks from the incompleteness, but also demonstrates the barrier effect diversity for different moneyness options. The NPI method is more closed to the reality, which investors can refer the results to avoid making arbitrary decisions under the complete market assumption.

## 4. Exotic option with non-monotonic payoffs: look-back option

We have implemented the NPI method to the barrier option, a relatively complicated type of exotic options with the monotonic payoff. In this section, the application of the NPI method to the exotic option with the non-monotonic payoff is illustrated by studying the look-back option.

'Look-back option' as one of the exotic option is introduced by Goldman, Sosin and Gatto [11]. The look-back option is classed into two types: the look-back option with the fixed strike price and the look-back option with the floating strike price. The option with fixed strike price entitles the option holder to get the payoff that is the difference between the maximum underlying asset price over the observation option period $\max_{0 \leq i \leq m} S(t_i)$ and the strike price $K_c$, $\max_{0 \leq i \leq m} S(t_i) - K_c$, for the call option or the difference between the strike price $K_p$ and the minimum underlying asset price during this period $\min_{0 \leq i \leq m} S(t_i)$, $K_p - \min_{0 \leq i \leq m} S(t_i)$, for the put option, where $i$ represents the time indicator between the initial time and maturity $m$. The one with a floating strike price gives the option holder the right of buying the underlying asset at the minimum underlying asset price during the option life period $\min_{0\leq i\leq m} S(t_i)$ or selling the underlying asset at its maximum price during this period $\max_{0\leq i\leq m} S(t_i)$. Goldman, Sosin, and Gatto [11] provided the pricing method based on the Brownian motion when they first presented this type of option. The CRR model can be used in the look-back option as well. Hull and White [15] elaborated the path-dependent option evaluation based on the binomial tree in 1993. In the same year, Amin[1] considered the generalization of the CRR model to make it suitable for path-dependent options' evaluation by adding a jump-diffusion process. Kima, Park and Qian [16] derived a binomial tree model with jump diffusion specific for the look-back option. Babbs [4] monitored the look-back option with a discrete time scheme instead of the continuous monitor based on the binomial tree. Park [18] also explored a binomial tree model with double-exponential jumps and studied its convergence. According to the definition of the look-back option with the fixed or floating strike price, the payoff of the look-back option is pretty clear. However, the payoff of the look-back option are not monotone with the path structure anymore. The tree in Figure 9 gives an example to explains the monotonicity of the option payoff.

In Figure 9, there is a tree of the stock price, and at the last step, we also listed the maximum and the minimum stock prices of each movement path. The stock price starts from $S_0$, and at each time step, it will go either up by the factor $u$ or down by the factor $d$. Generally, the monotonicity of the option value highly depends on the multiplication value of movement factors, $ud > 1$, $ud = 1$, or $ud < 1$. For example, in this 4-period option tree the option payoff is not monotonic when $ud < 1$. The maximum stock price of the path $S_0 \to S_1^1 \to S_2^2 \to S_3^2 \to S_4^2$ is $S_4^2$, while the maximum stock price of the path $S_0 \to S_1^1 \to S_2^1 \to S_3^2 \to S_4^3$ is $S_2^1$. Since $ud < 1$, then $S_4^2 < S_2^1$. Yet, when it comes to the other path ends with the same maturity stock price $S_4^2$, $S_0 \to S_1^1 \to S_2^1 \to S_3^1 \to S_4^2$, the maximum stock price is $S_3^1$ which is higher than the maximum stock price of the path $S_0 \to S_1^1 \to S_2^1 \to S_3^2 \to S_4^3$ ending with $S_4^3$, which is equal to $S_2^1$. This causes the results

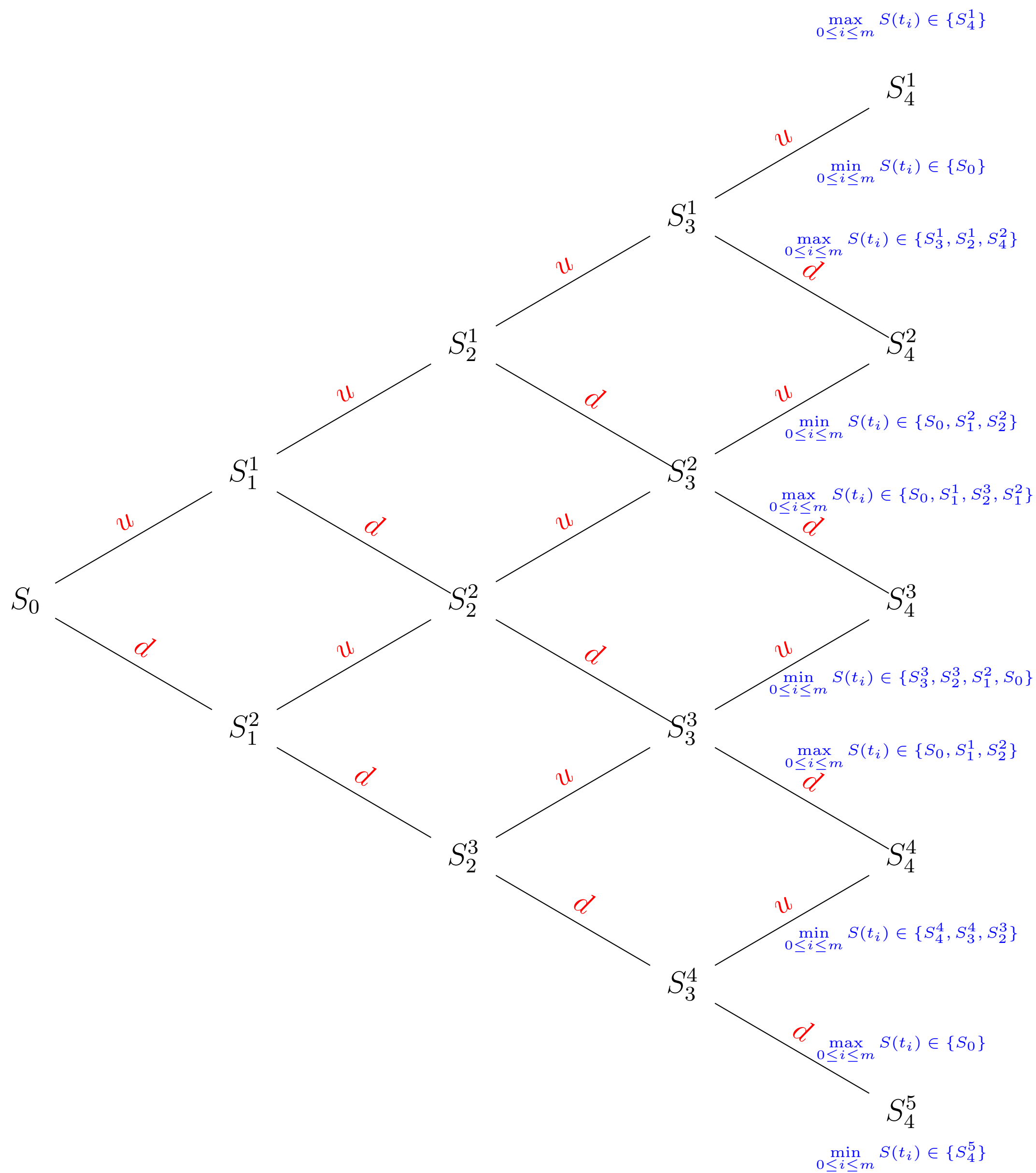


Figure 9: The binomial tree of the stock price with maximum and minimum stock price of each path

that look-back option payoff is not monotonic. So the option payoffs are not monotonic in the binomial tree. When the binomial tree is monotonic, then we can use the same NPI probability assignment as other types of options. If not, we need to think about the probability structure again, which is not considered in this paper but challenging and interesting topic for a future topic. Here instead of giving a new probability assignment, we offer a new binomial tree which is monotonic inspired by the look-back option pricing model presented by Cheuk and Vorst in 1997 [6].

Cheuk and Vorst [6] presented the new binomial approach for the look-back option with floating strike price. As acknowledged, the payoff of a look-back call option with a floating strike price is defined as $S(T) - \min_{0 \leq i \leq m} S(t_i)$. Here $t_0$ is the initial time of the option contract, and $t_m$ is the maturity time $T$. Then for any time in the binomial tree $t_j$, denote the minimum stock price of the option life period as $\underline{M}(t_j) = \min_{0 \leq i \leq j} S(t_i) = S(t_j)u^{-k}$, then the look-back call option value is $V(S(t_j), \underline{M}(t_j), t_j)$. Define the power of stock price upward movement factor $u$:

$$k = \ln[S(t_j)/\underline{M}(t_j)]/\ln(u) \tag{27}$$

$S(t_j) \geq \underline{M}(t_j)$, $k$ is positive integer and $k = 0, 1, \ldots, j$, so the option value at each time step can be transferred to a function depending on the stock price $S(t_j)$ and $k$, i.e.

$$V(S(t_j), \underline{M}(t_j), t_j) = S(t_j) - \underline{M}(t_j) = S(t_j)(1 - u^{-k}) = S(t_j)W_{t_j}(k) \tag{28}$$

This claim also holds for the maturity. Hence, by defining $W_{t_j}(k) = 1 - u^{-k}$ we can construct a new binomial tree of $W_{t_j}(k)$, $k = 0, 1, \ldots, j$.

In Figure 10, if $k \geq 1$ at $t_j$ and the stock price goes up to $S(t_{j+1}) = S(t_j)u$, then at time $t_{j+1}$ the power of $u$ is $k+1$. If the stock price goes down, the power of $u$ is $k-1$. While when $k = 0$, the situation is different, which for the upward movement the power of $u$ is 1, but the power of $u$ for downward movement is still 0. As we can see here the binomial tree of $W_{t_j}(k)$ with the path, we can use the NPI probabilities to evaluate the option. Here for

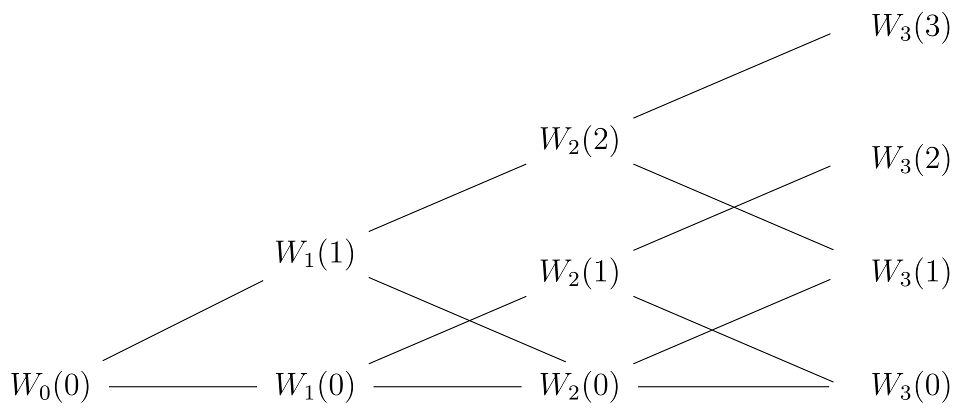


Figure 10: The lookback call option with the floating strike price

$n$ historical observations, and $s$ represents the number of times that the stock price went up in the previous time. Then we can get the upper and lower probabilities of upward movement from $W_{t_j}(k)$ to $W_{t_{j+1}}(k+1)$ as,

$$\overline{P}(t_j) = \frac{s+k+1}{n+t_j+1} \tag{29}$$

$$\underline{P}(t_j) = \frac{s+k}{n+t_j+1} \tag{30}$$

Then we apply these NPI probabilities to each one step path. Based on the NPI method, we compute the expected value of $W_0(0)$. By the definition of the look-back call option with the floating strike price, the option price is $S(0)W_0(0)$. The backward method for each node can be formulated as below.

The maximum buying price of the call option is

$$\begin{aligned} &\underline{V_0} = S(0)W_0(0) \\ &\underline{W}_{t_j}(k) = B(t_j, t_{j+1}) \left[ \underline{P}(t_j)\underline{W}_{t_{j+1}}(k+1) + (1-\underline{P}(t_j))\underline{W}_{t_{j+1}}(k-1) \right] \end{aligned} \tag{31}$$

The minimum selling price of the call option is

$$\overline{V_0} = S(0)W_0(0)$$
$$\overline{W_{t_j}}(k) = B(t_j, t_{j+1})\left[\overline{P}(t_j)\overline{W}_{t_{j+1}}(k+1) + (1-\overline{P}(t_j))\overline{W}_{t_{j+1}}(k-1)\right] \tag{32}$$

where $B(t_j, t_{j+1})$ is the discount factor from time $t_j$ to time $t_{j+1}$.

Similarly, we can construct the tree for the look-back put option with the floating payoff as well. By definition, the payoff of the look-back put option with the floating payoff is $\max_{0\le i\le m} S(t_i) - S(T)$, where $\max_{0\le i\le m} S(t_i)$ is the maximum stock price during the whole option life period. Then using a function to represent this value at time $t_j$ is $\overline{M}(t_j) = \max_{0\le i\le m} S(t_i) = S(t_j)u^{-k}$. The option value $V(S(t_j), \overline{M}(t_j), t_j)$ depends on three factors, stock price, maximum stock price and the time to maturity. Define the power of upward movement factor $u$:

$$k = \ln[S(t_j)/\overline{M}(t_j)]/\ln(u) \tag{33}$$

As we know, $S(t_j)$ is always less than or equal to $\overline{M}(t_j)$, then $k$ is a negative integer belongs to the set of values $\{0, \ldots, -j\}$. Then we can rewrite the option value as,

$$V(S(t_j), \overline{M}(t_j), t_j) = \max_{0\le i\le m} S(t_i) - S(t_j) = (u^{-k} - 1)S(t_j) = S(t_j)G_{t_j}(k) \tag{34}$$

Define $G_{t_j}(k) = u^{-k} - 1$, then we construct a new binomial tree of $G(k, t_j)$.

In Figure 11, we display the binomial tree of $G(k, t_j)$. When $k$ is negative and the stock price goes down at $t_j$, for the next time step the power of $u$ is $k-1$. Or if the stock price goes up at time $t_j$, the power of $u$ is $k+1$ at time $t_{j+1}$. When $k = 0$, the downward movement will change the $k$ to $k-1$, but upward movement won't change the power of $u$. With this monotonic tree, we can use the NPI method to calculate the maximum buying price and the minimum selling price of the option. Here $n$ is the number of the historical stock price, among them $s$ stock prices go down. Then for each downward path, we have

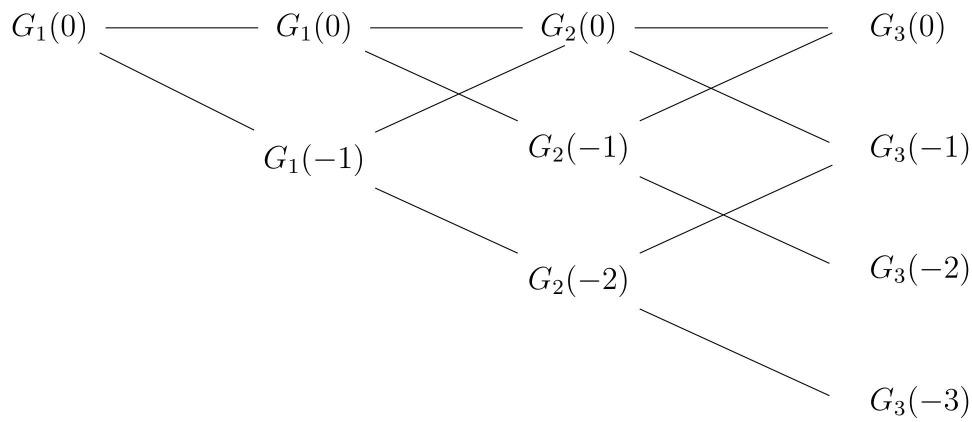


Figure 11: The lookback put option with the floating strike price

the upper and lower probabilities as:

$$\overline{P}(t_j) = \frac{s-k+1}{n+t_j+1} \tag{35}$$

$$\underline{P}(t_j) = \frac{s-k}{n+t_j+1} \tag{36}$$

The backward method for each node based on NPI leads to maximum buying price

$$\begin{aligned} &\underline{V_0} = S(0)G(0,0) \\ &\underline{G}_{t_j}(k) = B(t_j, t_{j+1}) \left[ (1-\underline{P}(t_j))\underline{G}_{t_{j+1}}(k+1) + \underline{P}(t_j)\underline{G}_{t_{j+1}}(k-1) \right] \end{aligned} \tag{37}$$

and minimum selling price

$$\begin{aligned} &\overline{V_0} = S(0)G(0,0) \\ &\overline{G}_{t_j}(k) = B(t_j, t_{j+1}) \left[ (1-\overline{P}(t_j))\overline{G}_{t_{j+1}}(k+1) + \overline{P}(t_j)\overline{G}_{t_{j+1}}(k-1) \right] \end{aligned} \tag{38}$$

where we first get the initial value of $G(k, t_j)$, $G(0,0)$, based on the backward method, then referring to the definition, we predict the option upper and lower prices $\overline{V_0}$ $\underline{V_0}$ by multiplying the inital stock price $S(0)$ and $G(0,0)$.

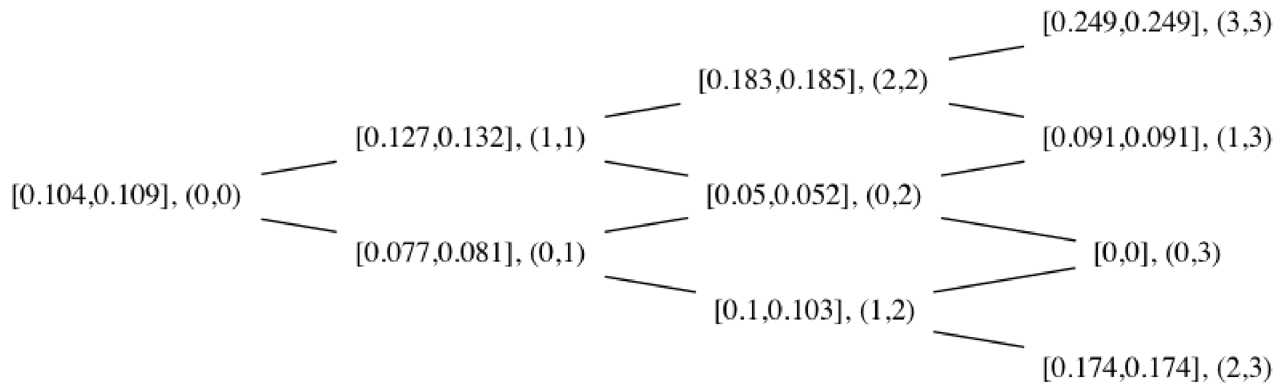


Figure 12: The binomial tree of a look-back call option with the floating strike price

**Example 4.1**

In this example, we use R program to predict the value of a look-back call option with the floating strike price $K = \min_{0 \le i \le m} S(t_i)$ derived from the stock with an initial stock price $S_0 = 20$. Following the mathematical description for the look-back call option, we first set up a binomial tree of $W_{t_j}(k)$ as shown in Figure 12. In the tree, there are four values at each node. Two values in the parenthesis are the power of the upward movement factor $k$ and the time steps $t$. The value outside the parenthesis is the lower and upper bounds of $W_{t_j}(k)$ at each node. After this discount backward evaluation method, we get the values $\underline{W}_0(0) = 0.104$ and $\overline{W}_0(0) = 0.109$, then we can calculate the maximum buying price and the minimum selling price of the look-back option, which are $\underline{V}(0) = S_0\underline{W}_0(0) = 20 \times 0.104 = 2.08$ and $\overline{V}(0) = S_0\overline{W}_0(0) = 20 \times 0.109 = 2.18$.

Since the look-back option with a floating strike price has a fixed moneyness, we compare the results from the NPI method and the CRR model with different life periods to take an insight into the NPI method for this kind of look-back option. Figure 13 shows the predictive results from both methods for look-back options with varying time to maturity based on 50 historical data and 25 upward movement historical data. In this study, the CRR results always locate in the interval provided by the NPI method. However, the interval from the NPI method gets wider along with the increasing time steps limited by the size of the available historical data. As the NPI method is a framework learning

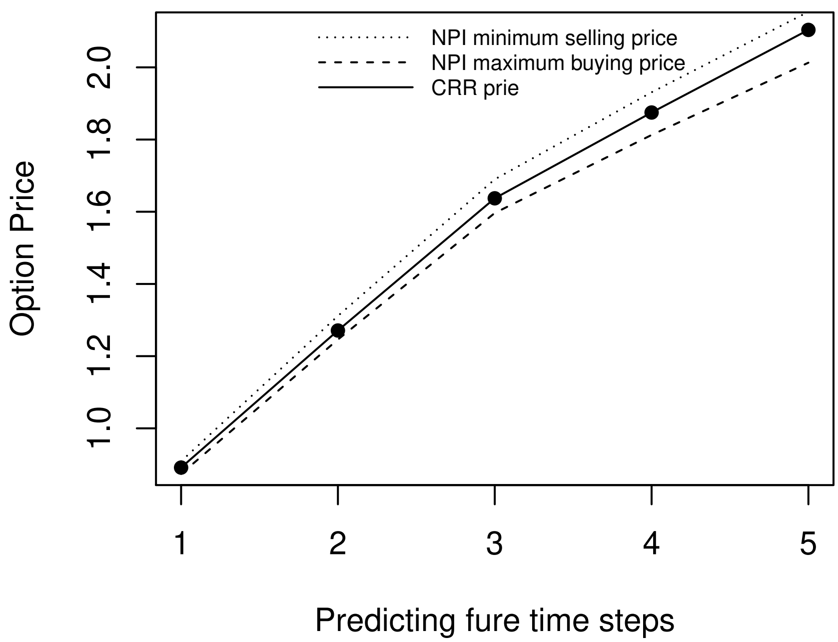


Figure 13: The Comparison between NPI and CRR with $n = 50$ and $s = 25$

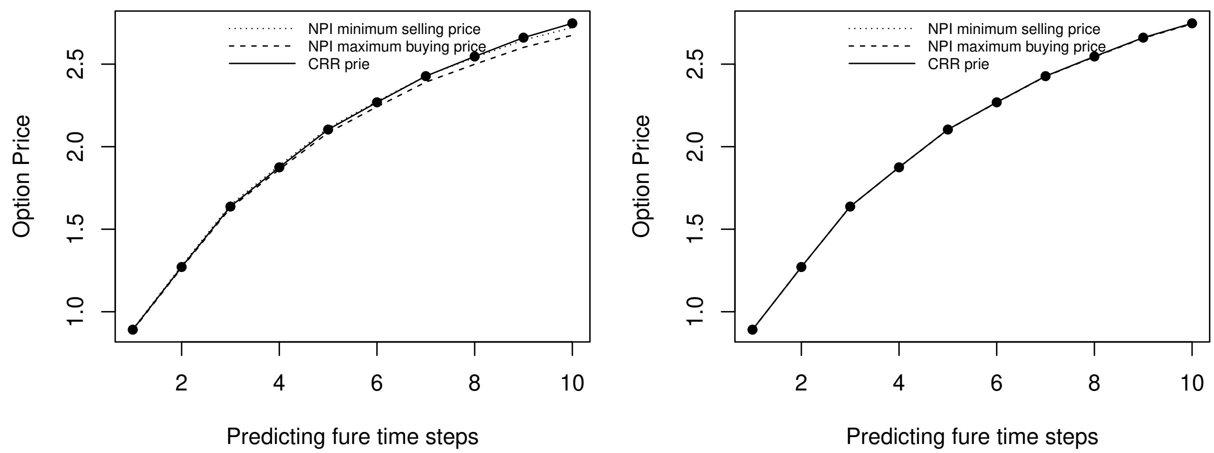


Figure 14: The Comparison between NPI and CRR with $n = 252,\ s = 126$ (left) $n = 5000,\ s = 2500$ (right)

the information fully from the historical data, the precision and accuracy of the result are highly upon the sufficiency of the historical data. For further study, we increase the number of historical data to 252 and 5000 keeping the same proportion of the increased historical data. The result is demonstrated in Figure 14. It is pretty obvious that by increasing the size of the historical data, the NPI result narrows down approaching to a precise value and finally identical to the result from the CRR model. Therefore, as long as the market is complete meaning that it delivers sufficient and correct information, the NPI method generates the same result as the CRR does under the assumption of the complete market.

## 5. Conclusion and Remarks

In this paper, we have presented the NPI method for exotic option pricing. Our method provides an evaluation price procedure that can be utilized when the market is not complete. As the NPI method is an imprecise probability inference approach learning and updating the information from the historical data, we illustrate its implementation in exotic option pricing with both monotonic payoffs and non-monotonic payoffs. We categorize exotic options by the payoff monotonicity and study one type of exotic option in each category, the barrier option and the look-back option.

For the barrier option, since this type of option has monotonic payoffs, the imprecise probabilities are assign to the binomial tree directly by using the NPI method. Formally, we present the backward optimization method for valuing the barrier option according to the NPI method. By comparing the results from the NPI method and the CRR model with different moneyness, the NPI method offers an interval price containing the CRR forecasting and the precision of the NPI prices is getting better when there are more historical data available in the market.

As the look-back option has non-monotonic payoffs in the binomial tree, we could not assign the imprecise probability directly to the binomial tree of the underling asset as was done for the barrier option. Inspired by the work done by Cheuk and Vorst [6], we manipulate the binomial tree according to the number of upward movements in future to guarantee the monotonicity and present the mathematical description of the pricing procedure based on our method. Besides, we investigate the performance of our method with examples. The outcomes display that for as long as the market offers sufficient information, our method can provide the precise result, and for the incomplete market, our method can also provide a reasonable result.

Overall, the main advantage of our method is that it provides a formal way to compute and represent the price of the exotic option with few assumptions and reflects more uncertainty even though the market is incomplete with limited information. When the market is complete, our method also leads to a precise option price for exotic options.

In this paper, we price exotic options with non-monotonic payoffs by manipulating the binomial tree. Alternatively, a new imprecise probability assignment measure can be derived by the definition of the option as an interesting and challenging way of solution. Future study for our approach could be the investigation of the NPI method in the real market by applying the empirical data and comparing the forecasting result with the real market price.

## Funding

The first-named author acknowledges the funding from Capital University of Economics and Business as Fundamental Research Funds for the Beijing Universities (XRZ2020042).

## Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study

## Aknowledgement

The authors are grateful to two reviewers of this article for for detailed comments that have resulted in improved presentation.

## References

[1] K.I. Amin. Jump diffusion option valuation in discrete time. *The Journal of Finance*, 48:1833–1863, 1993.

[2] E. Appolloni, M. Gaudenzi, and A. Zanette. The binomial interpolated lattice method for step double barrier options. *International Journal of Theoretical and Applied Finance*, 17:1450035, 2014.

[3] T. Augustin and F.P.A. Coolen. Nonparametric predictive inference and interval probability. *Journal of Statistical Planning and Inference*, 124:251–272, 2004.

[4] S. Babbs. Binomial valuation of lookback options. *Journal of Economic Dynamics and Control*, 24:1499–1525, 2000.

[5] P.P. Boyle and S.H. Lau. Bumping up against the barrier with the binomial method. *The Journal of Derivatives*, 1:6–14, 1994.

[6] T.H.F. Cheuk and T.C.F. Vorst. Currency lookback options and observation frequency: a binomial approach. *Journal of International Money and Finance*, 16:173–187, 1997.

[7] F.P.A. Coolen. Low structure imprecise predictive inference for bayes' problem. *Statistics and Probability Letters*, 36:349–357, 1998.

[8] J. Cox and M. Rubinstein. *Options Markets*. Prentice–Hall, New Jersey, 1985.

[9] J.C. Cox, S.A. Ross, and M. Rubinstein. Option pricing: A simplified approach. *Journal of Financial Economics*, 7:229–263, 1979.

[10] M. Gaudenzi and M.A. Lepellere. Pricing and hedging american barrier options by a modified binomial method. *International Journal of Theoretical and Applied Finance*, 9:533–553, 2006.

[11] M. Goldman, H. Sosin, and M. Gatto. Path dependent options: Buy at the low, sell at the high. *Journal of Finance*, 34:1111–1128, 1979.

[12] T. He, F.P.A. Coolen, and T. Coolen-Maturi. Nonparametric predictive inference for european option pricing based on the binomial tree model. *Journal of the Operational Research Society*, 70:1692–1780, 2019.

[13] T. He, F.P.A. Coolen, and T. Coolen-Maturi. Nonparametric predictive inference for american option pricing based on the binomial tree model. *Communication in Statistics-Theory and Methods*, 2020.

[14] B.M. Hill. Posterior distribution of percentiles: Bayes' theorem for sampling from a population. *Journal of the American Statistical Association*, 63:677–691, 1968.

[15] J Hull and A. White. Efficient procedures for valuing european and american path-dependent options. *Journal of Derivatives*, 1(1):21–31, 1993.

[16] K.I. Kim, H.S. Park, and X. Qian. A mathematical modeling for the lookback option with jump–diffusion using binomial tree method. *Journal of Computational and Applied Mathematics*, 235:5140–5154, 2011.

[17] R.C. Merton. Theory of rational option pricing. *The Bell Journal of Economics and Management Science*, 4:141–183, 1973.

[18] H.S. Park. Analytical binomial lookback options with double-exponential jumps. *Journal of the Korean Statistical Society*, 38:397–404, 2009.

[19] M. Reimer and K. Sandmann. A discrete time approach for european and american barrier options. *Discussion Paper, Murray State University*, SFB 303(No. B-272), 1996.

[20] M. Rubinstein. Exotic Options. `http://www.haas.berkeley.edu/groups/finance/WP/rpf220.pdf`, 1990.

[21] M. Rubinstein and E. Reiner. Breaking down the barriers. *RISK*, 4:28–35, 1991.