Nonparametric Predictive Inference for Two Future Observations with Right-Censored Data

Tahani Coolen-Maturi[a,*], Ali M. Mahnashi[b], Frank P. A. Coolen[a]

[a]*Department of Mathematical Sciences, Durham University, Durham, DH1 3LE, United Kingdom*
[b]*Department of Mathematics, College of Science, Jazan University, Jazan, Saudi Arabia*

Abstract

In reliability and survival analyses, right-censored observations are common. This type of data occurs when an event of interest is not fully observed during an experiment and there is no information provided about a random quantity, except that it exceeds a certain value. Nonparametric Predictive Inference (NPI) is a frequentist statistical method that relies on only few assumptions. It quantifies uncertainty by using imprecise probabilities based on Hill's assumption $A_{(n)}$ and focuses specifically on future observations. NPI has been developed for various types of data, including right-censored data, for some inferences such as multiple group comparisons, uncertainty quantification of the survival function, and in the context of competing risks. However, NPI with right-censored data has only considered a single future observation. This paper aims to extend this method by considering two future observations and taking into account that in the NPI approach, such multiple future observations are not conditionally independent given the data. Specifically, we present NPI lower and upper probabilities for the event that both future observations are greater than a particular time. Examples are provided for illustration and an application to system reliability is presented.

Keywords: Nonparametric predictive inference, right-censored data, censoring, imprecise probability, future observations, system reliability.

---

1. Introduction

In survival analysis, one of the primary characteristics is that some data may not be fully observable, but are instead censored. In many cases, event times are subject to right-censoring, which simply means that for a specific individual it is known that the event has not yet occurred at a particular time [1]. In other words, an observation for an individual is right-censored at c if its lifetime is only known to be greater than c. While there are several other common types of censoring, including left-censoring and interval-censoring, right-censoring occurs most frequently in applied statistics. This paper considers data sets including right-censored observations.

Nonparametric Predictive Inference (NPI) is a frequentist statistical method that is based on Hill's assumption $A_{(n)}$ [2, 3], which uses imprecise probabilities [4, 5, 6, 7, 8] to quantify

*Tahani Coolen-Maturi, email: tahani.maturi@durham.ac.uk

uncertainty. NPI gives lower and upper probabilities for a future, observable, random quantity, conditioned on observed values of related random quantities, based on the assumption $A_{(n)}$ [9]. NPI has been developed for a variety of data types, such as Bernoulli data [10, 11], real-valued data [12, 13, 14], data with right-censored observations [15, 16], bivariate data [17], multinomial data [18, 19], and circular data [20]. Moreover, NPI has been developed for a wide variety of statistical applications, such as reliability analysis [21], operational research [22] and medical survival data [23]. This paper is mostly theoretical in nature, and uses data from the literature to illustrate how the developed methods are used.

Due to the fact that the NPI method is a frequentist method based on Hill's assumption $A_{(n)}$ and utilizes the imprecise probability theory to quantify uncertainty, in this section, we will discuss the nature and properties of $A_{(n)}$ as well as some basic aspects of imprecise probability theory. Assume that $X_1, X_2, \dots, X_n, X_{n+1}$ are real-valued absolutely continuous and exchangeable random quantities. Let the ordered observed values of $X_1, \dots, X_n$ be denoted as $x_1 < x_2 < \dots < x_n$. To simplify notation, let $x_0 = -\infty$ and $x_{n+1} = \infty$, or we assume $x_0 = 0$ in case of nonnegative random quantities [24]. It is assumed that there are no ties between the observations of the data. In the case of ties, we assume that the tied observations differ by a small amount, which is a common strategy in statistics to break ties [3]. These n observations divide up the real-line into $n+1$ intervals $I_j = (x_j, x_{j+1})$, where $j = 0, 1, \dots, n$. Based on n observations, the assumption $A_{(n)}$ [25] is that the probability that the next future observation X_{n+1} is equally likely to fall in each open interval (x_j, x_{j+1}), for all $j = 0, 1, \dots, n$, so

$$P_{X_{n+1}}(x_j, x_{j+1}) = \frac{1}{n+1} \qquad \text{for all } \ j = 0, 1, \dots, n \tag{1}$$

The data carry information about the location but no information about the rank of the future observations, corresponding to the absence of prior knowledge, so $A_{(n)}$ is considered as a post-data assumption related to finite exchangeability, and assumes nothing else [26]. For a detailed presentation and discussion of $A_{(n)}$, see Hill [25].

The assumption $A_{(n)}$ alone is insufficient for constructing precise probabilities for many events of interest, but it is still useful to derive bounds for probability, effectively by applying De Finetti's Fundamental Theorem of Probability [26], or Walley's concept of natural extension [5], which provide lower and upper probabilities in interval probability theory. Weichselberger [27] also developed a formal foundation for interval probability, via lower and upper probabilities, by applying the principles of Kolmogorov's axioms. These lower and upper probabilities are also known as imprecise probabilities in accordance with the imprecise probability theory [9, 8].

Imprecise probabilities have been proposed and studied since at least the middle of the nineteenth century [28]. Recently, the topic of imprecise probabilities has become increasingly prominent, resulting in a series of conferences and a project website, *www.sipta.org*. There are several interpretations of the lower and upper probabilities for event A, which are denoted by $\underline{P}(A)$ and $\overline{P}(A)$, respectively [20]. According to Walley [5], for instance, the lower and upper probabilities for event A can be interpreted as supremum buying price and infimum selling price, respectively, of a gamble on the event A, in which 1 is paid when the event occurs and 0 if the event does not occur. From a classical perspective, lower and upper probabilities can be interpreted as bounds on precise probabilities, because of the lack of information or the desire not to make further assumptions. The theory of imprecise probability clearly demonstrates that bounds provide valuable information regarding the uncertainty of events caused by a lack of information [9, 5, 7, 29, 27]. The precise classical probability of an event A is simply a special case of the imprecise probability, when $\underline{P}(A) = \overline{P}(A)$, whereas the total absence of information about the event A can be reflected by $\underline{P}(A) = 0$ and $\overline{P}(A) = 1$. Next, we outline several important aspects of imprecise probability theory relevant to $A_{(n)}$-based inference [9]. As a general rule, in imprecise probability theory, the lower and upper probabilities for the event A are $\underline{P}(A) = 1 - \overline{P}(A^c)$, which is the conjugacy property, where A^c represents the complementary event of A. In many cases, this conjugacy property can be utilised in order to simplify the calculation of imprecise probabilities for events of interest and their complementary events. For events A and B, such that $A \cap B = \emptyset$, the lower probability is superadditive and the upper probability is subadditive, that is

$$\underline{P}(A \cup B) \geq \underline{P}(A) + \underline{P}(B) \text{ and } \overline{P}(A \cup B) \leq \overline{P}(A) + \overline{P}(B)$$

In the following section, we will introduce the statistical method NPI which assigns lower and upper probabilities to events involving a future random observation X_{n+1}.

Furthermore, Coolen and Yan [15] have developed NPI for right-censored data based on a generalization of $A_{(n)}$, called the right-censoring $A_{(n)}$ assumption, or rc-$A_{(n)}$, but it was only developed for a single future observation. In practice, however, there may be reasons to be interested in multiple future observations; it is important that in the NPI approach, such multiple future observations are not conditionally independent given the data. In this paper, we develop NPI for two future observations based on the assumption rc-$A_{(n)}$ without further assumptions and as an example application we consider reliability of series systems.

This rest of the paper is organised as follows. First a brief overview about NPI for right-censored data is given in Section 2. Using a new approach, we reformulate the NPI lower and upper probabilities for the event $X_{n+1} > t$ in Section 3. In Section 4, we present NPI for the event $X_{n+2} > t$ given $X_{n+1} > t$. NPI for the joint event $X_{n+1} > t$ and $X_{n+2} > t$ is presented in Section 5. Section 6 illustrates how these inferences can be applied to quantify the reliability of a small series system. Finally, this paper ends with concluding remarks in Section 7. For further details and discussion, we would like to direct the reader to the thesis of the second author [30].

2. NPI for right-censored data

Hill's assumption $A_{(n)}$ [31] by itself is not suitable for right-censored data, so Coolen and Yan [15] presented a generalization of $A_{(n)}$, called the right-censoring $A_{(n)}$ assumption, abbreviated as rc-$A_{(n)}$, for right-censored data. They added a new assumption to $A_{(n)}$ to makes it more suitable for dealing with right-censored data. It is assumed that, at the moment of censoring, the residual lifetime of a right-censored observation is exchangeable with the residual lifetimes of all other observations that are not yet failed or censored [15].

According to the $A_{(n)}$ assumption [31], the probability distribution for a real-valued random quantity X_{n+1} is partially specified by probability masses assigned to open intervals, without any further restriction on the spread of the probability mass within each interval [15, 32]. A probability mass assigned in such a way to an interval (a, b) is denoted by

$M_X(a,b)$, and referred to as a M-function value for $X \in (a,b)$. The M-function value should satisfy $0 \leq M_X(a,b) \leq 1$ and the M-function values specified on all intervals should sum up to one [15]. These M-functions are also in the theory presented by Shafer [33].

In this section, we follow the notation and definitions presented by Maturi [34]. Consider the following data when determining the predictive probabilities for a future observation. Assume $X_1, \ldots, X_n, X_{n+1}$ are non-negative, exchangeable and continuous random quantities representing lifetimes. Suppose that there are in total n observations containing u failure times observations, $x_1 < x_2 < \cdots < x_u$, and $\nu = n - u$ right-censoring times, $c_1 < c_2 < \cdots < c_\nu$. For ease of notation, $x_0 = 0$ and $x_{u+1} = \infty$. Suppose further that there are s_i right-censored observations in the interval $I^i = (x_i, x_{i+1})$, denoted by $c_1^1 < c_2^i < \cdots < c_{s_i}^i$, so $\sum_{i=}^{u} s_i = \nu$, such that $c_{i^*}^i \in (x_i, x_{i+1})$, where $i = 0, 1, \ldots, u$ and $i^* = 1, 2, \ldots, s_i$. Assuming there are no ties between the data observations, the issue of dealing with ties has been previously discussed by Maturi [34].

On the basis of n given event times, the assumption $A_{(n)}$ offers a partially specified probability distribution for X_{n+1} in terms of M-function values. To deal with right-censored observations being present in the data, a generalization of $A_{(n)}$ was considered, that is the assumption $\tilde{A}_{(n)}$ [15].

Definition 2.1. ($\tilde{A}_{(n)}$ assumption)
On the basis of data including u event times and $\nu = n - u$ right-censoring times, the assumption $\tilde{A}_{(n)}$ partially specifies the probability distribution for the next observation X_{n+1} assigning probability masses to two types of open intervals, one formed by consecutive event times, (x_i, x_{i+1}), and the other is formed by a censoring time and infinity, $(c_{i^*}^i, \infty)$, expressed via the following M-function values:

$$\tilde{M}_{X_{n+1}}(x_i, x_{i+1}) = \frac{1}{n+1} \tag{2}$$

$$\tilde{M}_{X_{n+1}}(c_{i^*}^i, \infty) = \frac{1}{n+1} \tag{3}$$

where $i = 0, 1, \ldots, u$ and $i^* = 1, 2, \ldots, s_i$.

Note that the notation $\tilde{M}$ used in Equations (2) and (3) indicates that the values of the M-function are based on the assumption of $\tilde{A}_{(n)}$. Based on Equation (2), the probability masses for the intervals (x_i, x_{i+1}) formed by the event times u are equal to $\frac{1}{n+1}$. Moreover, without any additional assumptions, a probability mass of $\frac{1}{n+1}$ is assigned to the interval $(c_{i^*}^i, \infty)$, and as per Equation (3), the lifetime of this observation will occur at any point past $c_{i^*}^i$. Finally, the probability mass assigned to the interval $(c_{i^*}^i, \infty)$ is divided into masses on sub-intervals, as described in [15].

Let $X_{c_{i^*}^i}$ denote the random quantity corresponding to the right-censoring at time $c_{i^*}^i$. According to [15], the probability masses assigned to intervals $(c_{i^*}^i, \infty)$ may have caused wide bounds on probabilities, so it would be helpful if these probability masses can be split into probability masses on sub-intervals. For this reason, Coolen and Yan [15, 35] proposed the assumption Shifted-$\tilde{A}_{(n)}$ for $X_{c_{i^*}^i}$, for which all we know is that the random quantity $X_{c_{i^*}^i}$

exceeds $c^i_{i^*}$.

Definition 2.2. (Shifted-$\tilde{A}_{(n)}$ assumption)
The assumption shifted-$\tilde{A}_{(n)}$ partially specifies the probability distribution for $X_{c^i_{i^*}}$, given that $X_{c^i_{i^*}} > c^i_{i^*}$, expressed via the following M-function values:

$$M_{X_{c^i_{i^*}}}(x_k, x_{k+1}) = \frac{1}{\tilde{n}_{c^i_{i^*}} + 1} \qquad \text{for } \; k = i+1, \ldots, u, \tag{4}$$

$$M_{X_{c^i_{i^*}}}(c^i_{i^*}, x_{k+1}) = \frac{1}{\tilde{n}_{c^i_{i^*}} + 1}, \tag{5}$$

$$M_{X_{c^i_{i^*}}}(c^i_l, \infty) = \frac{1}{\tilde{n}_{c^i_{i^*}} + 1} \qquad \text{for } \; l = i^*+1, \ldots, \nu. \tag{6}$$

where $\tilde{n}_{c^i_{i^*}}$ represents the number of observations in the risk set at time $c^i_{i^*}$, for $c^i_{i^*} \in (x_i, x_{i+1})$, $i^* = 1, 2, \ldots, s_i$.

This assumption is related to the fact that if the random quantities $X_1, X_2, \ldots, X_r$ are exchangeable, then the random quantities in any subset of $X_1, X_2, \ldots, X_r$ are also exchangeable [15, 35]. It also follows that as long as the random quantities $X_1, X_2, \ldots, X_r$ are exchangeable, then all are also exchangeable when they exceed a given value c [15, 35]. In this sense, the exchangeability assumption of all random quantities known to be in the risk set just prior to c_i is an appropriate assumption to handle random quantities that are right-censored at time $c^i_{i^*}$, and in fact implies the assumption of non-informative censoring [15, 35].

Based on the assumption of non-informative censoring, the assumption shifted-$\tilde{A}_{(n)}$ allows us to apply $A_{(n)}$ but with the starting point shifted from the value 0 to the observed right-censoring time $c^i_{i^*}$ [15, 35]. Clearly, the sum of the M-function values for $X_{c^i_{i^*}}$ over these sub-intervals, as in Equations (4), (5) and (6), is equal to one [15, 35].

Taking into account the two previously proposed assumptions, $\tilde{A}_{(n)}$ for X_{n+1} and 'shifted-$\tilde{A}_{(n)}$ for $X_{c^i_{i^*}}$, Coolen and Yan [15, 35] proposed the right-censoring $\tilde{A}_{(n)}$ assumption, denoted by rc-$\tilde{A}_{(n)}$, which allows splitting the total M-function values for X_{n+1} assigned to interval $(c^i_{i^*}, \infty)$ into separate M-function values for X_{n+1} assigned to sub-intervals of $(c^i_{i^*}, \infty)$.

Definition 2.3. (rc-$\tilde{A}_{(n)}$ assumption)
Let $\mathcal{P}_{c^i_{i^*}} = M_{X_{n+1}}(c^i_{i^*}, \infty)$ be the M-function value for X_{n+1} on the interval $(c^i_{i^*}, \infty)$, taking into account the effects of all previous right-censorings and $A_{(n)}$. The assumption rc-$\tilde{A}_{(n)}$

splits the probability mass of $M_{X_{n+1}}(c^i_{i^*}, \infty)$ as

$$M^{c^i_{i^*}}_{X_{n+1}}(x_k, x_{k+1}) = \frac{\mathcal{P}_{c^i_{i^*}}}{\tilde{n}_{c^i_{i^*}} + 1} \qquad \text{for } \; k = i+1, \ldots, u, \tag{7}$$

$$M^{c^i_{i^*}}_{X_{n+1}}(c^i_{i^*}, x_{k+1}) = \frac{\mathcal{P}_{c^i_{i^*}}}{\tilde{n}_{c^i_{i^*}} + 1}, \tag{8}$$

$$M^{c^i_{i^*}}_{X_{n+1}}(c^i_l, \infty) = \frac{\mathcal{P}_{c^i_{i^*}}}{\tilde{n}_{c^i_{i^*}} + 1} \qquad \text{for } \; l = i^*+1, \ldots, \nu. \tag{9}$$

where $\tilde{n}_{c^i_{i^*}}$ represents the number of observations in the risk set at time $c^i_{i^*}$, for $c^i_{i^*} \in (x_i, x_{i+1})$, where $i = 0, 1, \ldots, u$ and $i^* = 1, 2, \ldots, s_i$.

With the combined assumptions $\tilde{A}_{(n)}$ and rc-$\tilde{A}_{(n)}$ for $r = 1, 2, \ldots, i^*-1$, $i^* = 1, 2, \ldots, s_i$, and for any right-censoring time $c^i_{i^*}$, the $\mathcal{P}_{c^i_{i^*}}$ can be computed by

$$\mathcal{P}_{c^i_{i^*}} = M_{X_{n+1}}(c^i_{i^*}, \infty) = \frac{1}{n+1} \prod_{\{r : r < i^*\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}}. \tag{10}$$

where $\tilde{n}_{c_r}$ is the number of individuals in the risk set just prior to time c_r [15, 35]. Note that throughout this paper, a product over an empty set is defined to be equal to 1.

Consequently, based on the assumptions $\tilde{A}_{(n)}$, given by Definition 2.1, and rc-$\tilde{A}_{(n)}$, given by the Definition 2.3, the M-function values for X_{n+1} are finally all assigned to intervals (x_i, x_{i+1}) or $(c^i_{i^*}, x_{i+1})$ for $i = 0, 1, \ldots, u$ and $i^* = 1, 2, \ldots, s_i$, via considering an assumption called right-censoring $A_{(n)}$, which is also denoted as rc-$A_{(n)}$ [15, 35].

Definition 2.4. (rc-$A_{(n)}$ assumption)
The assumption rc-$A_{(n)}$ partially specifies the NPI-based probability distribution for the observable and non-negative random quantity X_{n+1}, via the following M-function values [32],

$$M_{X_{n+1}}(x_i, x_{i+1}) = \frac{1}{n+1} \prod_{\{r : c_r < x_i\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} \tag{11}$$

$$M_{X_{n+1}}(c^i_{i^*}, x_{i+1}) = \frac{1}{(n+1)\tilde{n}_{c^i_{i^*}}} \prod_{\{r : c_r < c^i_{i^*}\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} \tag{12}$$

where $i = 0, 1, \ldots, u$, $i^* = 1, 2, \ldots, s_i$ and $\tilde{n}_{c_r}$ represents the number of observations in the risk set just before time c_r.

Following Maturi [34] and based on the assumption rc-$A_{(n)}$, all M-function values that are assigned for X_{n+1} to be in one interval created by two consecutive observed event times,

(x_i, x_{i+1}), lead to the following probability for the event $X_{n+1} \in (x_i, x_{i+1})$,

$$\begin{aligned} P_{X_{n+1}}(x_i, x_{i+1}) &= M_{X_{n+1}}(x_i, x_{i+1}) + \sum_{i^*=1}^{s_i} M_{X_{n+1}}(c^i_{i^*}, x_{i+1}) \\ &= \frac{1}{n+1} \prod_{r: c_r < x_{i+1}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} \end{aligned} \tag{13}$$

Based on the rc-$A_{(n)}$ assumption, Maturi [34] derived simple closed-form expressions for the NPI lower and upper survival functions, $\underline{S}_{X_{n+1}}(t) = \underline{P}(X_{n+1} > t)$ and $\overline{S}_{X_{n+1}}(t) = \overline{P}(X_{n+1} > t)$. Coolen and Yan [15] compared the NPI lower and upper survival functions based on the rc-$A_{(n)}$ assumption with the Kaplan–Meier estimator. They showed that the lower survival function for X_{n+1}, based on the assumption rc-$A_{(n)}$, becomes zero after the largest observation, also the KM estimator will behave this way if that observation is an event time. The upper survival function always remains positive, unless the range of possible values for X_{n+1} is restricted by choosing a finite upper bound [15, 35]. The KM estimate is always equal to one for the first interval $(0, x_1)$, which is the case for the NPI upper survival function. It is worth mentioning that the KM estimate only decreases at observed event times. The NPI lower survival function decreases at every observation but the NPI upper survival function decreases only at event times, like the KM. Coolen and Yan [15] claimed that the rc-$A_{(n)}$-based lower and upper survival functions for X_{n+1} are more suitable for graphical presentation compared to the KM-based lower and upper survival functions, as they show the data in full, including right-censored observations, and can be interpreted in a predictive manner [15, 35].

Coolen and Yan [15] presented NPI for right-censored data for a single future observation. There is a challenge to generalise the approach of NPI for right-censored data to multiple future observations. NPI has been developed to multiple future observations for uncensored real-valued data [36, 37] and for Bernoulli data [10], however, this is complicated for right-censored data. In this paper, further theory is developed on NPI for two future observations with attention to right-censored data. This will be achieved by applying the rc-$A_{(n)}$ assumption [15], without further assumptions, for X_{n+1} and, conditionally on X_{n+1}, applying the rc-$A_{(n+1)}$ assumption for X_{n+2}. The focus is on NPI lower and upper probabilities for the event that both future observations X_{n+1} and X_{n+2} are greater than time t. We also illustrate how the proposed method can be applied to system reliability.

3. Reformulating NPI for the first future observation

The main objective of this paper is to develop NPI for two future observations, X_{n+1} and X_{n+2}, for data that involves right-censored observations. Particularly, we present NPI lower and upper probabilities for the event $X_{n+1} > t$ and $X_{n+2} > t$. According to the rc-$A_{(n)}$ [15] assumption, the probability distribution for X_{n+1} is partially specified by probability mass assigned to open nested intervals via M-function values, without further restrictions on where it is in each interval. We consider X_{n+1} and X_{n+2} such that X_{n+2} is conditioned on X_{n+1} and the data set that contains n observations with right-censored observations. Without making any further assumptions, we aim to apply the rc-$A_{(n)}$ [15] assumption for X_{n+1}, and then, conditionally on X_{n+1}, we will apply the rc-$A_{(n+1)}$ assumption for X_{n+2}. However, determining where to allocate probability masses for specific events of interest to obtain lower and upper probabilities for the NPI can be challenging. So, we must consider where the probability mass is for X_{n+1} within an interval (x_i, x_{i+1}), in order to apply rc-$A_{(n+1)}$ for X_{n+2}. In this case, this interval (x_i, x_{i+1}), which contains right-censored observations, must be specified into sub-intervals $(c^i_{i^*}, x_{i+1})$, $i^* = 1, 2, \dots, s_i$, with respect to that the probability mass for X_{n+1} according to its M-function value assigned to the interval (x_i, x_{i+1}), will be distributed over these sub-intervals $(c^i_{i^*}, x_{i+1})$. To do this, we introduce probabilities denoted by $\underline{\alpha}^i$ and $\underline{\alpha}^{c^i_{i^*}}$, $i = 0, 1, \dots, u$ and $i^* = 1, 2, \dots, s_i$, to enable us to determine where to put the probability mass per interval over its sub-intervals. In this way, we can minimise or maximise the probability for any event of interest involving the one or two future observations with regard to the $\underline{\alpha}^i$ and $\underline{\alpha}^{c^i_{i^*}}$ values. Overall, this allows deriving the NPI lower and upper probabilities for the event $X_{n+1} > t$.

To this ends, we start with deriving the lower and upper probabilities for the event $X_{n+1} > t$, which has been done by Coolen and Yan [15], in a different way. For an interval $I^i = (x_i, x_{i+1})$, $i = 0, 1, 2, \dots, u$, there are s_i right-censored observations in this interval, and

$$\underline{\alpha}^i = (\alpha^i_1, \alpha^i_2, \dots, \alpha^i_{s_i+1}), \text{where } 0 \leq \alpha^i_{i^*} \leq 1 \text{ and } \sum_{i^*=1}^{s_i+1} \alpha^i_{i^*} = 1$$

If there are no censored observations in the interval (x_i, x_{i+1}), that is $s_i = 0$, then $\underline{\alpha}^i = \alpha^i_1 = 1$. And, for each censored observation $c^i_{i^*}$, $i^* = 1, 2, \dots, s_i$, in the interval (x_i, x_{i+1}),

$$\underline{\alpha}^{c^i_{i^*}} = (\alpha_1^{c^i_{i^*}}, \alpha_2^{c^i_{i^*}}, \dots, \alpha_{s_i-i^*+1}^{c^i_{i^*}}), \text{where } 0 \leq \alpha_l^{c^i_{i^*}} \leq 1 \text{ and } \sum_{l=1}^{s_i-i^*+1} \alpha_l^{c^i_{i^*}} = 1.$$

If there is only one censored observation in the interval (x_i, x_{i+1}) then $\underline{\alpha}^{c^i_{i^*}} = \alpha_1^{c^i_{i^*}} = 1$.

The notation $\underline{\alpha}^i$ and $\underline{\alpha}^{c^i_{i^*}}$ are the proportion of (a specific) probability mass assigned to the intervals (x_i, x_{i+1}) and (c^i_l, x_{i+1}), respectively, that are distributed over sub-intervals. It is just a way to specify how the probability mass is divided over sub-intervals, so that we can then find the NPI lower and upper probabilities for any event of interest involving X_{n+1}. The $\alpha^i_{i^*}$ are introduced to determine where to place the probability mass per interval (x_i, x_{i+1}) over its sub-intervals, whereas the $\alpha_l^{c^i_{i^*}}$ are introduced to determine where to place the probability mass per interval (c^i_l, x_{i+1}) over its sub-intervals.

Consequently, we reformulate the original M-function masses shown in Definition 2.4, having the notation $\underline{\alpha}^i$ introduced to them, to specify how much of each M-function value is in sub-intervals.

Definition 3.1. (rc-$A_{(n)}$- revisited)

Let $I^i_{i^*} = (t^i_{i^*}, t^i_{i^*+1})$ represent an interval created by the n data observations, where $i = 0, 1, 2, \dots, u$, and

$$\begin{cases} i^* = 0 & \text{if} \quad t^i_0 = x_i \quad \text{(failure time or time 0)} \\ i^* = 1, 2, \dots s_i, & \text{if} \quad t^i_{i^*} = c^i_{i^*} \quad \text{(right-censoring time)} \end{cases}$$

and for simplicity of notation let $t^i_{s_i+1} = t^{i+1}_0 = x_{i+1}$. Thus, the assumption rc-$A_{(n)}$ partially specifies the NPI-based probability distribution for the observable, non-negative and real-valued random quantity X_{n+1}, via the following M-function values.

$$M_{X_{n+1}}(t^i_{i^*}, t^i_{i^*+1}) = \alpha^i_{i^*+1} M_{X_{n+1}}(x_i, x_{i+1}) + \sum_{l=1}^{i^*} \alpha^{c^i_l}_{i^*-l+1} M_{X_{n+1}}(c^i_l, x_{i+1}) \tag{14}$$

In Equation (14), the M-function values $M_{X_{n+1}}(x_i, x_{i+1})$ and $M_{X_{n+1}}(c^i_{i^*}, x_{i+1})$ are derived from Equations (11) and (12), respectively. The $M_{X_{n+1}}(t^i_{i^*}, t^i_{i^*+1})$, stated in Equation (14), could be also denoted by $M_{X_{n+1}\in I^i_{i^*}}$. We do this for convenience in order to be used later in Section 5.

With respect to that for all $\alpha^i_{i^*} \in [0,1]$, $\alpha^{c^i_{i^*}}_l \in [0,1]$, $\sum_{i^*=1}^{s_i+1} \alpha^i_{i^*} = 1$ and $\sum_{l=1}^{s_i-i^*+1} \alpha^{c^i_{i^*}}_l = 1$, the M-function values as specified by rc-$A_{(n)}$ in Definition 3.1 lead to the probability for the event that $X_{n+1} \in (x_i, x_{i+1})$, $i = 0, 1, \ldots, u$, denoted by $P_{X_{n+1}}(x_i, x_{i+1})$, which can be calculated by summing up all M-function values assigned to the interval $I^i = (x_i, x_{i+1})$ along with all M-function values assigned to them sub-intervals $(c^i_{i^*}, x_{i+1})$ for X_{n+1}, so that

$$\begin{aligned}
P_{X_{n+1}}(x_i, x_{i+1}) &= \sum_{i^*=0}^{s_i} M_{X_{n+1}}(t^i_{i^*}, t^i_{i^*+1}) \\
&= \sum_{i^*=0}^{s_i} \alpha^i_{i^*+1} M_{X_{n+1}}(x_i, x_{i+1}) + \sum_{i^*=1}^{s_i} \sum_{l=1}^{i^*} \alpha^{c^i_l}_{i^*-l+1} M_{X_{n+1}}(c^i_l, x_{i+1}) \\
&= M_{X_{n+1}}(x_i, x_{i+1}) + \sum_{l=1}^{s_i} \sum_{i^*=1}^{s_i-l+1} \alpha^{c^i_l}_{i^*} M_{X_{n+1}}(c^i_l, x_{i+1}) \\
&= M_{X_{n+1}}(x_i, x_{i+1}) + \sum_{l=1}^{s_i} M_{X_{n+1}}(c^i_l, x_{i+1})
\end{aligned} \tag{15}$$

for $i = 0, 1, \ldots, u$. As expected, Equation (15) is identical to Equation (13). For convenience, $P_{X_{n+1}}(x_i, x_{i+1})$, stated in Equation (15), will be also denoted by $P_{X_{n+1}\in I^i}$. The first term after the second equality in Equation (15) is the sum of all M-function values assigned to the interval (x_i, x_{i+1}), and as $\sum_{i^*=1}^{s_i+1} \alpha^i_{i^*} = 1$, this first term is equal to $M_{X_{n+1}}(x_i, x_{i+1})$. The second term after the second equality in Equation (15) is the sum of all M-function values assigned to the sub-intervals (c^i_l, x_{i+1}) of (x_i, x_{i+1}), and as $\sum_{l=1}^{s_i-i^*+1} \alpha^{c^i_{i^*}}_l = 1$, for $i = 0, 1, \ldots, u$ and $i^* = 1, 2, \ldots, s_i$, this second term is equal to $\sum_{l=1}^{s_i} M_{X_{n+1}}(c^i_l, x_{i+1})$. Let us define the following equation

$$Q_{X_{n+1}}(t^i_a, x_{i+1}) = \sum_{i^*=a}^{s_i} \alpha^i_{i^*+1} M_{X_{n+1}}(x_i, x_{i+1}) + \sum_{i^*=a}^{s_i} \sum_{l=1}^{i^*} \alpha^{c^i_l}_{i^*-l+1} M_{X_{n+1}}(c^i_l, x_{i+1}) \tag{16}$$

where for $a = 0$, Equation (15) and (16) are equivalent. This Equation (16) can be minimised or maximised in order to derive the NPI lower and upper probabilities for the event $X_{n+1} > t$.

For convenience, we may refer to the probability in Equation (16) as $Q_{X_{n+1}\in I_a^i}$. This notation will be useful in Section 5.

Now, let us consider the second term of Equation (16), and by rearranging the summations, we have

$$\sum_{i^*=a}^{s_i}\sum_{l=1}^{i^*}\alpha_{i^*-l+1}^{c_l^i}M_{X_{n+1}}(c_l^i,x_{i+1}) = \sum_{l=1}^{a-1}\sum_{i^*=a}^{s_i-l+1}\alpha_{i^*}^{c_l^i}M_{X_{n+1}}(c_l^i,x_{i+1})+\sum_{l=a}^{s_i}\sum_{i^*=1}^{s_i-l+1}\alpha_{i^*}^{c_l^i}M_{X_{n+1}}(c_l^i,x_{i+1}) \tag{17}$$

The first term on the right-hand side of Equation (17) is related to the probability masses to the right of t_a^i, corresponding to all $c_l^i < t_a^i$. The second term in Equation (17) is related to the probability masses corresponding to all $c_l^i \geq t_a^i$, and as $\sum_{i^*=1}^{s_i-l+1}\alpha_{i^*}^{c_l^i} = 1$, this second term is equal to $\sum_{l=a}^{s_i} M_{X_{n+1}}(c_l^i, x_{i+1})$. So Equation (16) can be rewritten as

$$\begin{aligned} Q_{X_{n+1}}(t_a^i,x_{i+1}) = &\sum_{i^*=a}^{s_i}\alpha_{i^*+1}^{i}M_{X_{n+1}}(x_i,x_{i+1})+\sum_{l=1}^{a-1}\sum_{i^*=a}^{s_i-l+1}\alpha_{i^*}^{c_l^i}M_{X_{n+1}}(c_l^i,x_{i+1}) \\ &+\sum_{l=a}^{s_i}M_{X_{n+1}}(c_l^i,x_{i+1}) \end{aligned} \tag{18}$$

To determine the values of $\underline{\alpha}^i$ and $\underline{\alpha}^{c_l^i}$ that will minimize $Q_{X_{n+1}}(t_a^i, x_{i+1})$ as shown in Equation (18), we need to allocate all probability masses within the interval (x_i, x_{i+1}) to the left of t_a^i, that is

$$\sum_{i^*=a}^{s_i}\alpha_{i^*+1}^{i} = 0\ , \qquad \sum_{i^*=0}^{a-1}\alpha_{i^*+1}^{i} = 1$$

and

$$\sum_{i^*=1}^{a-1}\alpha_{i^*}^{c_l^i} = 1\ , \qquad \sum_{i^*=a}^{s_i-l+1}\alpha_{i^*}^{c_l^i} = 0$$

thus, the minimum value of $Q_{X_{n+1}}(t_a^i, x_{i+1})$ is

$$Q_{X_{n+1}}^{\min}(t_a^i,x_{i+1}) = \sum_{l=a}^{s_i}M_{X_{n+1}}(c_l^i,x_{i+1}) \tag{19}$$

Now, to determine the optimal values of $\underline{\alpha}^i$ and $\underline{\alpha}^{c_l^i}$ that maximize $Q_{X_{n+1}}(t_a^i, x_{i+1})$, as stated in Equation (18), we need to assign all probability masses in the interval (x_i, x_{i+1}) to the right of t_a^i, that is

$$\sum_{i^*=a}^{s_i}\alpha_{i^*+1}^{i} = 1\ , \qquad \sum_{i^*=0}^{a-1}\alpha_{i^*+1}^{i} = 0$$

and

$$\sum_{i^*=1}^{a-1}\alpha_{i^*}^{c_l^i} = 0\ , \qquad \sum_{i^*=a}^{s_i-l+1}\alpha_{i^*}^{c_l^i} = 1$$

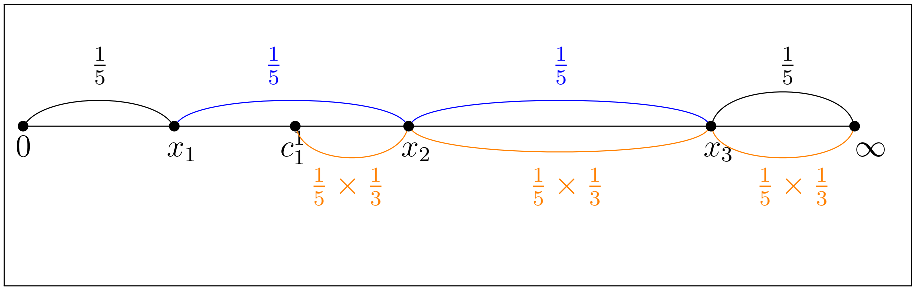

Figure 1: The original M-functions based on rc-$A_{(n)}$ assumption for X_5, Example 3.1

thus, the maximum value of $Q_{X_{n+1}}(t_a^i, x_{i+1})$ is

$$\begin{aligned} Q^{\max}_{X_{n+1}}(t_a^i, x_{i+1}) &= M_{X_{n+1}}(x_i, x_{i+1}) + \sum_{l=1}^{a-1} M_{X_{n+1}}(c_l^i, x_{i+1}) + \sum_{l=a}^{s_i} M_{X_{n+1}}(c_l^i, x_{i+1}) \\ &= M_{X_{n+1}}(x_i, x_{i+1}) + \sum_{l=1}^{s_i} M_{X_{n+1}}(c_l^i, x_{i+1}) \\ &= P_{X_{n+1}}(x_i, x_{i+1}) \end{aligned} \tag{20}$$

For ease of notation, we will refer to the probabilities $Q^{\min}_{X_{n+1}}(t_a^i, x_{i+1})$ and $Q^{\max}_{X_{n+1}}(t_a^i, x_{i+1})$ presented in Equations (19) and (20) as $Q^{\min}_{X_{n+1} \in I_a^i}$ and $Q^{\max}_{X_{n+1} \in I_a^i}$, respectively.

Consequently, the NPI lower probability for the event $X_{n+1} > t$, for $t \in [t_a^i, t_{a+1}^i)$ with $i = 0, 1, \ldots, u$ and $a = 0, 1, \ldots, s_i$, can be written as follows.

$$\begin{aligned} \underline{P}(X_{n+1} > t) &= Q^{\min}_{X_{n+1}}(t_{a+1}^i, x_{i+1}) + \sum_{j=i+1}^{u} P_{X_{n+1}}(x_j, x_{j+1}) \\ &= \sum_{l=a+1}^{s_i} M_{X_{n+1}}(c_l^i, x_{i+1}) + \sum_{j=i+1}^{u} P_{X_{n+1}}(x_j, x_{j+1}) \end{aligned} \tag{21}$$

The corresponding NPI upper probability for the event $X_{n+1} > t$, for $t \in [x_i, x_{i+1})$ with $i = 1, 2, \ldots, u$ and $a = 0, 1, \ldots, s_i$, can be written as follows.

$$\begin{aligned} \overline{P}(X_{n+1} > t) &= Q^{\max}_{X_{n+1}}(t_a^i, x_{i+1}) + \sum_{j=i+1}^{u} P_{X_{n+1}}(x_j, x_{j+1}) \\ &= \sum_{j=i}^{u} P_{X_{n+1}}(x_j, x_{j+1}) \end{aligned} \tag{22}$$

Example 3.1. Suppose that a data set consists of three failure observations at times x_1, x_2, x_3 and one right-censored observation at time c_1^1, as shown in Figure 1.

First, let us briefly illustrate the assumption rc-$A_{(n)}$ [15]. Let $X_{c_1^1}$ denote the random quantity corresponding to the right-censoring at time c_1^1, where $c_1^1 \in (x_1, x_2)$. According to the $\tilde{A}_{(4)}$ assumption, given by Definition 2.1, the M-function values for X_5 are

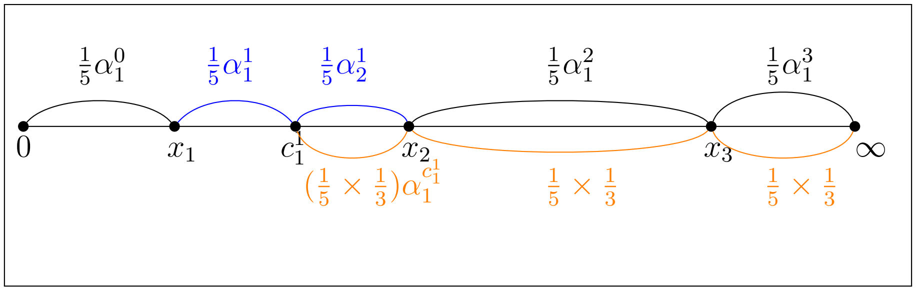

Figure 2: Reformulating the original M-functions for X_5, Example 3.1.

$\tilde{M}_{X_5}(0,x_1) = \tilde{M}_{X_5}(x_1,x_2) = \tilde{M}_{X_5}(x_2,x_3) = \tilde{M}_{X_5}(x_3,\infty) = \frac{1}{5}$, and a further probability mass 1/5 is distributed over the interval (c_1^1,∞), i.e. $\tilde{M}_{X_5}(c_1^1,\infty) = \frac{1}{5}$, since it is known without making any further assumptions that X_5 will be at any point beyond c_1^1.

As per the non-informative censoring assumption, the residual lifetime of the censored observation is independent of the censoring process, therefore, the assumption shifted-$\tilde{A}_{(2)}$, given by Definition 2.2, allows us to apply $A_{(2)}$ with the starting point shifted from 0 to the censoring time c_1^1. Based on the assumption shifted-$\tilde{A}_{(2)}$, the probability distribution for $X_{c_1^1}$, given $X_{c_1^1} > c_1^1$, is partially specified via M-function values for $X_{c_1^1}$ assigned to sub-intervals as $M_{X_{c_1^1}}(c_1^1,x_2) = M_{X_{c_1^1}}(x_2,x_3) = M_{X_{c_1^1}}(x_3,\infty) = \frac{1}{3}$. Moreover, the assumption rc-$\tilde{A}_{(4)}$, given by Definition 2.3, splits the probability mass of $\tilde{M}_{X_5}(c_1^1,\infty) = \frac{1}{5}$ to M-function values for X_5 assigned to sub-intervals as $M_{X_5}^{c_1^1}(c_1^1,x_2) = M_{X_5}^{c_1^1}(x_2,x_3) = M_{X_5}^{c_1^1}(x_3,\infty) = \frac{1}{5}\times\frac{1}{3} = \frac{1}{15}$.

The M-function values for X_5 based on the assumption $\tilde{A}_{(4)}$, given by Definition 2.1, are then combined with the M-function values for X_5 based on the assumption rc-$\tilde{A}_{(4)}$, given by Definition 2.3, leading to the M-function values for X_5 based on the rc-$A_{(4)}$ assumption, as given by the Definition 2.4 [15, 35]. For example, the M-function value for the event $X_5 \in (x_2,x_3)$ based on the assumption rc-$A_{(4)}$ is derived as $M_{X_5}(x_2,x_3) = \tilde{M}_{X_5}(x_2,x_3) + M_{X_5}^{c_1^1}(x_2,x_3) = \frac{1}{5} + \frac{1}{15} = \frac{4}{15}$.

Thus, the original M-function values for the first future observation X_5, based on the assumption rc-$A_{(n)}$ [15], according to the Definition 2.4, are (see also Figure 1),

$$
\begin{aligned}
M_{X_5}(0,x_1) &= \frac{1}{5} = \frac{3}{15}\\
M_{X_5}(x_1,x_2) &= \frac{1}{5} = \frac{3}{15}\\
M_{X_5}(c_1^1,x_2) &= \frac{1}{5}\times\frac{1}{3} = \frac{1}{15}\\
M_{X_5}(x_2,x_3) &= \frac{1}{5} + \frac{1}{15} = \frac{4}{15}\\
M_{X_5}(x_3,\infty) &= \frac{1}{5} + \frac{1}{15} = \frac{4}{15}.
\end{aligned}
$$

With the new technique presented in Section 3 on the basis of Definition 3.1, we have the opportunity to specify the original M-function values for X_5, shown in Figure 1, to probability mass values assigned to their sub-intervals, as shown in Figure 2.

From Figure 2, as the data set presented in this example does not include any censored observations in the intervals $I^0 = (0, x_1), I^2 = (x_2, x_3)$ and $I^3 = (x_3, \infty)$, we have $\alpha_1^0 = \alpha_1^2 = \alpha_1^3 = 1$. The interval $I^1 = (x_1, x_2)$ contains a single censored observation c_1^1, so we split this interval into two sub-intervals; $I_0^1 = (x_1, c_1^1)$ and $I_1^1 = (c_1^1, x_2)$ and we introduce α_1^1 and α_2^1 for these intervals, respectively, such that the sum of them is one. Using these α_1^1 and α_2^1 values, we can determine the distribution of a probability per interval over its sub-intervals in order to minimise or maximise the probability for the event $X_5 > t$.

As for $c_1^1 \in (x_1, x_2)$, it is necessary to determine where to put the probability mass for X_5, that is, $M_{X_5}(x_1, x_2) = \frac{1}{5}$, in this interval. Since there is only one right-censored observation in (x_1, x_2), the probability mass $M_{X_5}(x_1, x_2) = \frac{1}{5}$, given by Equation (7), is now assigned into two sub-intervals, with regard to α_1^1 and α_2^1 introduced respectively to the two sub-intervals, as

$$M_{X_5}(x_1, c_1^1) = \alpha_1^1 M_{X_5}(x_1, x_2) = \frac{1}{5}\alpha_1^1 \tag{23}$$

$$M_{X_5}(c_1^1, x_2) = \alpha_2^1 M_{X_5}(x_1, x_2) = \frac{1}{5}\alpha_2^1 \tag{24}$$

Taking into consideration the probability mass $M_{X_5}^{c_1^1}(c_1^1, x_2) = \frac{1}{15}$, given by Definition 2.3, we consider the following probability mass, using Definition 3.1, to be assigned to the sub-interval (c_1^1, x_2) for $c_1^1 \in (x_1, x_2)$

$$M_{X_5}(c_1^1, x_2) = \alpha_1^{c_1^1} M_{X_5}^{c_1^1}(c_1^1, x_2) = \frac{1}{15}\alpha_1^{c_1^1}, \qquad \text{where } \alpha_1^{c_1^1} = 1 \tag{25}$$

Therefore, the original M-function values for X_5 [15], given by the Definition 2.4 and shown in Figure 1, are now re-distributed based on the Definition 3.1, as follow (see Figure 2),

$$\begin{aligned}
M_{X_5}(0, x_1) &= \frac{1}{5} \\
M_{X_5}(x_1, c_1^1) &= \frac{1}{5}\alpha_1^1 \\
M_{X_5}(c_1^1, x_2) &= \frac{1}{5}\alpha_2^1 + \frac{1}{15}\alpha_1^{c_1^1} \\
M_{X_5}(x_2, x_3) &= \frac{1}{5} + \frac{1}{15} \\
M_{X_5}(x_3, \infty) &= \frac{1}{5} + \frac{1}{15}
\end{aligned}$$

Then, for the interval (x_1, x_2) which contains the only right-censored observation c_1^1, we consider $Q_{X_5}(c_1^1, x_2)$ as representing a probability that can either be maximised or minimised depending on how much the probability mass value is distributed over the sub-intervals of the interval (x_1, x_2). Using Equation (16), the function $Q_{X_5}(c_1^1, x_2)$ is defined by combining Equations (24) and (25), as

$$\begin{aligned}
Q_{X_5}(c_1^1, x_2) &= \alpha_2^1 M_{X_5}(x_1, x_2) + \alpha_1^{c_1^1} M_{X_5}^{c_1^1}(c_1^1, x_2) \\
&= \frac{1}{5}\alpha_2^1 + \frac{1}{15}\alpha_1^{c_1^1}
\end{aligned}$$

$t \in (.)$	$\underline{P}(X_5 > t)$	$\overline{P}(X_5 > t)$
$(0, x_1)$	$\frac{4}{5}$	1
(x_1, c_1^1)	$\frac{3}{5}$	$\frac{4}{5}$
(c_1^1, x_2)	$\frac{8}{15}$	$\frac{4}{5}$
(x_2, x_3)	$\frac{4}{15}$	$\frac{8}{15}$
(x_3, ∞)	0	$\frac{4}{15}$

Table 1: $\underline{P}(X_5 > t)$ and $\overline{P}(X_5 > t)$ according to Example 3.1

but $\alpha_1^{c_1^1} = 1$ since there is only one right-censored observation in the interval (x_1, x_2), so $Q_{X_5}(c_1^1, x_2) = \frac{1}{5}\alpha_2^1 + \frac{1}{15}$.

The function $Q_{X_5}(c_1^1, x_2)$ can be minimised and maximised in order to obtain the NPI lower and upper probabilities for the event $X_5 \in (c_1^1, x_2)$, using Equations (19) and (20). The minimum value of the function $Q_{X_5}(c_1^1, x_2)$ is obtained by assigning all probability masses within the interval (x_1, x_2) to the left of c_1^1, that is $\alpha_2^1 = 0$, so $\alpha_1^1 = 1$ and $Q_{X_5}^{\min}(c_1^1, x_2) = \frac{1}{15}$. The maximum value of the function $Q_{X_5}(c_1^1, x_2)$ is obtained by assigning all probability masses within the interval (x_1, x_2) to the right of c_1^1, that is $\alpha_2^1 = 1$, so $\alpha_1^1 = 0$ and $Q_{X_5}^{\max}(c_1^1, x_2) = \frac{1}{5} + \frac{1}{15} = \frac{4}{15}$.

The NPI lower and upper probabilities for the event $X_5 > t$, based on the Definition 3.1, are derived using Equations (21) and (22) respectively. The lower probability $\underline{P}(X_5 > t)$ is obtained by considering only the probability mass that necessarily lies in (t, ∞). The corresponding upper probability $\overline{P}(X_5 > t)$ is obtained by considering the probability mass that could possibly lie within (t, ∞).

Taking the case $t \in (x_1, c_1^1)$ as an example, the lower probability for the event $X_5 > t$ is obtained by considering only probability masses that necessarily lie within (t, ∞), using Equation (21), i.e., $\underline{P}_{X_5}(x_1, c_1^1) = Q_{X_5}^{\min}(c_1^1, x_2) + P_{X_5}(x_2, x_3) + P_{X_5}(x_3, \infty) = 1/15 + 4/15 + 4/15 = 3/5$. For the case $t \in (c_1^1, x_2)$, the upper probability for the event $X_5 > t$ is obtained by summing up all probability masses that can be in (t, ∞), using Equation (22), i.e., $\overline{P}_{X_5}(c_1^1, x_2) = Q_{X_5}^{\max}(c_1^1, x_2) + P_{X_5}(x_2, x_3) + P_{X_5}(x_3, \infty) = 4/15 + 4/15 + 4/15 = 4/5$. For t in an interval which does not contain right-censored observations, the NPI lower and upper probabilities for the event $X_5 > t$ can be derived directly from the closed-form expressions derived by Maturi [34]. Consequently, the NPI lower and upper probabilities for the event $X_5 > t$, based on the data in this example, are given in Table 1.

Note that we can straightforwardly apply rc-$A_{(4)}$ for X_5, using Definition 2.4, where there are no assumptions on where the probability mass is within each interval. But, as we aim to apply rc-$A_{(5)}$ for X_6, based on rc-$A_{(4)}$ for X_5, later on, we had to consider where the probability mass is for X_5 in this example using the new techniques presented in Section 3.

Next we need to consider the second future observation, X_{n+2}, based on the first future observation, X_{n+1}, as well as the data set that includes n observations with right-censored observations. Section 4 derives the NPI lower and upper conditional probabilities for $X_{n+2} > t$ given $X_{n+1} > t$, which will enable us to derive the NPI lower and upper probabilities for the event that both future observations are greater than t, in Section 5.

4. Lower and upper probabilities for $X_{n+2} > t$ given $X_{n+1} > t$

In this section, we will provide the NPI conditional lower and upper probabilities for the event $X_{n+2} > t$ given that $X_{n+1} > t$. To do this, we will use the rc-$A_{(n+1)}$ assumption for X_{n+2}, which applied conditionally on X_{n+1}. Additionally, we will apply the rc-$A_{(n)}$ assumption to X_{n+1}, which was previously explained in Section 3.

Based on Definition 3.1, there are $n+1$ cases of which X_{n+1} falls in the intervals created by the data set that contains n observations including right-censored observations, denoted as $I^i_{i*} = (t^i_{i^*}, t^i_{i^*+1})$, where $i = 0, 1, \ldots, u$, $i^* = 1, 2, \ldots, s_i$. For $X_{n+1} \in (t^i_{i^*}, t^i_{i^*+1})$, when considering X_{n+2}, there will be $n+1$ observations of which we have $u+1$ event times, $x_1 < x_2 < \cdots < x_u < x_{u+1}$, and $\nu = (n+1) - (u+1) = n - u$ right-censored observations, $c_1 < c_2 < \ldots < c_\nu$. Note that $u+1$ refer to the failure observations in the data set including X_{n+1}. So, there are $n+2$ intervals created by the data set that contains $n+1$ observations, included X_{n+1}, and the right-censored observations, denoted by $I^j_{j*} = (t^j_{j^*}, t^j_{j^*+1})$, where $j = 0, 1, \ldots, u+1$, $j^* = 1, 2, \ldots, s_j$. Let $x_0 = 0$ and $x_{u+2} = \infty$ for ease of notation. We assume, in order to simplify our presentation, that no ties exist in the data set, so no two observations (events or right-censoring) are at the same time value. In case there are ties, we refer to the discussion in [34].

To derive the NPI conditional lower and upper probabilities for the event $X_{n+2} > t$ given $X_{n+1} > t$, we will introduce the rc-$A_{(n+1)}$ assumption for X_{n+2} given $X_{n+1} \in I^i_{i*} = (t^i_{i^*}, t^i_{i^*+1})$. This follows the approach outlined in Section 3 for the rc-$A_{(n)}$ assumption for X_{n+1}, where we consider the probability mass of X_{n+1} within an interval (x_i, x_{i+1}). Here, in case of the event $X_{n+2} > t$ given $X_{n+1} > t$, we use the same notation that used for the event $X_{n+1} > t$ in Section 3, with replacing the notation $\underline{\alpha}^i$ and $\underline{\alpha}^{c^i_{i^*}}$ by $\underline{\beta}^i$ and $\underline{\beta}^{c^i_{i^*}}$.

Given that $X_{n+1} \in I^i_{i*} = (t^i_{i^*}, t^i_{i^*+1})$ and for an interval $I^j = (x_j, x_{j+1})$, $j = 0, 1, 2, \ldots, u+1$, there are s_j right-censored observations in this interval, and

$$\underline{\beta}^j = (\beta^j_1, \beta^j_2, \ldots, \beta^j_{s_j+1}), \text{where } 0 \leq \beta^j_{j^*} \leq 1 \text{ and } \sum_{j^*=1}^{s_j+1} \beta^j_{j^*} = 1$$

If there are no censored observations in the interval (x_j, x_{j+1}), that is $s_j = 0$, then $\underline{\beta}^j = \beta^j_1 = 1$. Also, for each censored observation $c^j_{j^*}$, $j^* = 1, 2, \ldots, s_j$, in the interval (x_j, x_{j+1}),

$$\underline{\beta}^{c^j_{j^*}} = (\beta^{c^j_{j^*}}_1, \beta^{c^j_{j^*}}_2, \ldots, \beta^{c^j_{j^*}}_{s_j-j^*+1}), \text{where } 0 \leq \beta^{c^j_{j^*}}_l \leq 1 \text{ and } \sum_{l=1}^{s_j-j^*+1} \beta^{c^j_{j^*}}_l = 1.$$

if there is only one censored observation in the interval (x_j, x_{j+1}) then $\underline{\beta}^{c^j_{j^*}} = \beta^{c^j_{j^*}}_1 = 1$.

The notation $\underline{\beta}^j$ and $\underline{\beta}^{c^j_{j^*}}$ are the proportion of a (specific) conditional probability mass assigned to the interval (x_j, x_{j+1}) that is distributed over sub-intervals, given $X_{n+1} \in (x_j, x_{j+1})$. It is just a way to write how the probability mass is divided over sub-intervals, so that we can then find the NPI conditional lower and upper probabilities for any event of interest involving X_{n+2} given X_{n+1}.

Given that $X_{n+1} \in I^i_{i*} = (t^i_{i^*}, t^i_{i^*+1})$, the rc-$A_{(n+1)}$ assumption partially specifies the probability distribution for the second future observation X_{n+2} by the conditional M-functions denoted as $M_{X_{n+2}|X_{n+1}}$. We present the conditional M-functions for X_{n+2} to be in the interval $I^j_{j*} = (t^j_{j^*}, t^j_{j^*+1})$, $j = 0, 1, \dots, u+1$, $j^* = 1, 2, \dots, s_j$, given that X_{n+1} is in the interval $I^i_{i*} = (t^i_{i^*}, t^i_{i^*+1})$, by the following definition.

Definition 4.1. (Conditional M-functions)
The conditional M-function partially specifies the probability distribution for the second future observation X_{n+2} given $X_{n+1} \in I^i_{i^*}$, i.e., $x_{n+1} \in (t^i_{i^*}, t^i_{i^*+1})$, for $i = 0, 1, \dots, u$, $i^* = 1, 2, \dots, s_i$, as follows

$$\begin{aligned} M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(t^j_{j^*}, t^j_{j^*+1}) &= \beta^j_{j^*+1} M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(x_j, x_{j+1}) \\ &\quad + \sum_{k=1}^{j^*} \beta^{c^j_k}_{j^*-k+1} M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(c^j_k, x_{j+1}) \end{aligned} \tag{26}$$

where

$$\begin{cases} j^* = 0 & \text{if} \quad t^j_0 = x_j \quad \text{(failure time or time 0)} \\ j^* = 1, 2, \dots s_j & \text{if} \quad t^j_{j^*} = c^j_{j^*} \quad \text{(right-censoring time)} \end{cases}$$

for $j = 0, 1, \dots, u+1$ and $j^* = 1, 2, \dots, s_j$, and for simplicity of notation let $t^j_{s_j+1} = t^{j+1}_0 = x_{j+1}$. For simplicity of notation, we can refer to $M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(t^j_{j^*}, t^j_{j^*+1})$ as $M_{X_{n+2}\in I^j_{j^*}|X_{n+1}\in I^i_{i^*}}$.

Similar to Definition 2.4, one can obtain the expression for the $M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(x_j, x_{j+1})$ and $M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(c^j_{j^*}, x_{j+1})$, given in Equation (26), as

$$M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(x_j, x_{j+1}) = \frac{1}{n+2} \prod_{\{r:c_r<x_j\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} \tag{27}$$

$$M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(c^j_{j^*}, x_{j+1}) = \frac{1}{(n+2)\tilde{n}_{c^j_{j^*}}} \prod_{\{r:c_r<c^j_{j^*}\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} \tag{28}$$

where $\tilde{n}_{c_r}$ represents the number of observations in the risk set (still functioning or alive and uncensored) just before time c_r. The product terms in Equations (27) and (28) are assumed to be equal to one if the product is taken over an empty set [15].

By utilising the M-functions for X_{n+1} according to Definition 3.1, we arrive at the expression for $P_{X_{n+1}}(x_i, x_{i+1})$ as shown in Equation (15). Similarly, if we use the conditional M-functions for $X_{n+2}|X_{n+1}$ based on Definition 4.1, we can obtain $P_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(x_j, x_{j+1})$ as given in Equation (29). With $\beta^j_{j^*} \in [0,1]$, $\beta^{c^j_{j^*}}_l \in [0,1]$, $\sum_{j^*=1}^{s_j+1} \beta^j_{j^*} = 1$ and $\sum_{l=1}^{s_j-j^*+1} \beta^{c^j_{j^*}}_l = 1$, the conditional M-function values as specified by rc-$A_{(n+1)}$ in Definition 4.1 lead to the conditional probability for the event that $X_{n+2} \in I^j_{j^*}$, where $j = 0, 1, \dots, u+1$, given $X_{n+1} \in I^i_{i^*}$, where $i = 0, 1, \dots, u$, denoted by $P_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(x_j, x_{j+1})$. The $P_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(x_j, x_{j+1})$ is calculated by summing up all conditional M-function values assigned to the interval $I^j = (x_j, x_{j+1})$ given $X_{n+1} \in I^i_{i^*}$, along with all conditional M-function values assigned to the sub-intervals $(c^j_{j^*}, x_{j+1})$ for X_{n+1}, given $X_{n+1} \in I^i_{i^*}$ so that

$$
\begin{aligned}
P_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(x_j, x_{j+1}) &= \sum_{j^*=0}^{s_j} M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(t^j_{j^*}, t^j_{j^*+1}) \\
&= \sum_{j^*=0}^{s_j} \beta^j_{j^*+1} M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(x_j, x_{j+1}) + \sum_{j^*=1}^{s_j}\sum_{l=1}^{j^*} \beta^{c^j_l}_{j^*-l+1} M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(c^j_l, x_{j+1}) \\
&= M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(x_j, x_{j+1}) + \sum_{l=1}^{s_j}\sum_{j^*=1}^{s_j-l+1} \beta^{c^j_l}_{j^*} M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(c^j_l, x_{j+1}) \\
&= M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(x_j, x_{j+1}) + \sum_{l=1}^{s_j} M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(c^j_l, x_{j+1}) \qquad (29)
\end{aligned}
$$

for $i = 0, 1, \ldots, u$ and $j = 0, 1, \ldots, u+1$. To simplify notation, we will refer to $P_{X_{n+2}|X_{n+1}\in I^i}(x_j, x_{j+1})$, in Equation (29), as $P_{X_{n+2}\in I^i|X_{n+1}\in I^i}$.

The first term after the second equality in Equation (29) is the sum of all the conditional M-function values assigned to the interval (x_j, x_{j+1}), given $X_{n+1} \in I^i_{i^*}$, and as $\sum_{j^*=1}^{s_j+1} \beta^j_{j^*} = 1$, this first term is equal to $M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(x_j, x_{j+1})$. The second term after the third equality in Equation (29) is the sum of all the conditional M-function values assigned to the sub-intervals (c^j_l, x_{j+1}) of (x_j, x_{j+1}), given $X_{n+1} \in I^i_{i^*}$, and as $\sum_{l=1}^{s_j-j^*+1} \beta^{c^j_{j^*}}_l = 1$, for $j = 0, 1, \ldots, u+1$ and $j^* = 1, 2, \ldots, s_j$, this second term is equal to $\sum_{l=1}^{s_j} M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(c^j_l, x_{j+1})$. And let us define the following

$$
\begin{aligned}
Q_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(t^j_a, x_{j+1}) = &\sum_{j^*=a}^{s_j} \beta^j_{j^*+1} M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(x_j, x_{j+1}) \\
&+ \sum_{j^*=a}^{s_j}\sum_{l=1}^{j^*} \beta^{c^j_l}_{j^*-l+1} M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(c^j_l, x_{j+1}) \qquad (30)
\end{aligned}
$$

where for $a = 0$, Equation (29) and (30) are equivalent.

The $Q_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(t^j_a, x_{j+1})$, given by Equation (30), can be minimised or maximised in order to derive the NPI conditional lower and upper probabilities for the event $X_{n+2} > t$ given $X_{n+1} > t$. We sometimes denote the conditional probability in Equation (30) by $Q_{X_{n+2}\in I^j_a|X_{n+1}\in I^i_{i^*}}$ for convenience. Now, let us consider the second term of Equation (30), and by rearranging the summations, we have

$$
\begin{aligned}
\sum_{j^*=a}^{s_j}\sum_{l=1}^{j^*} \beta^{c^j_l}_{j^*-l+1} M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(c^j_l, x_{j+1}) = &\sum_{l=1}^{a-1}\sum_{j^*=a}^{s_j-l+1} \beta^{c^j_l}_{j^*} M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(c^j_l, x_{j+1}) \\
&+ \sum_{l=a}^{s_j}\sum_{j^*=1}^{s_j-j+1} \beta^{c^j_l}_{j^*} M_{X_{n+2}|X_{n+1}\in I^i_{i^*}}(c^j_l, x_{j+1}) \qquad (31)
\end{aligned}
$$

The first term on the right-hand side of Equation (31) is related to the conditional probability masses to the right of t^j_a, corresponding to all $c^j_l < t^j_a$. The second term in Equation

(31) is related to the conditional probability masses corresponding to all $c_l^j \geq t_a^j$, and as $\sum_{j^*=1}^{s_j-j+1} \beta_{j^*}^{c_l^j} = 1$, this second term is equal to $\sum_{l=a}^{s_j} M_{X_{n+2}|X_{n+1}\in I*i_{i*}}(c_l^j, x_{j+1})$. So Equation (30) can be rewritten as

$$
\begin{aligned}
Q_{X_{n+2}|X_{n+1}\in I_{i^*}^i}(t_a^j, x_{j+1}) = &\sum_{j^*=a}^{s_j} \beta_{j^*+1}^j M_{X_{n+2}|X_{n+1}\in I_{i^*}^i}(x_j, x_{j+1}) \\
&+ \sum_{l=1}^{a-1} \sum_{j^*=a}^{s_j-l+1} \beta_{j^*}^{c_l^j} M_{X_{n+2}|X_{n+1}\in I_{i^*}^i}(c_l^j, x_{j+1}) \\
&+ \sum_{l=a}^{s_j} M_{X_{n+2}|X_{n+1}\in I_{i^*}^i}(c_l^j, x_{j+1})
\end{aligned} \tag{32}
$$

To determine the optimal values of $\underline{\beta}^j$ and $\underline{\beta}^{c_l^j}$ that minimize $Q_{X_{n+2}|X_{n+1}\in I_{i^*}^i}(t_a^j, x_{j+1})$, as presented in Equation (32), we need to allocate all conditional probability masses in (x_j, x_{j+1}) to the left of t_a^j, that is

$$
\sum_{j^*=a}^{s_j} \beta_{j^*+1}^j = 0 \ , \qquad \sum_{j^*=0}^{a-1} \beta_{j^*+1}^j = 1
$$

and

$$
\sum_{j^*=1}^{a-1} \beta_{j^*}^{c_l^j} = 1 \ , \qquad \sum_{j^*=a}^{s_j-l+1} \beta_{j^*}^{c_l^j} = 0
$$

thus, the minimum value of $Q_{X_{n+2}|X_{n+1}\in I_{i^*}^i}(t_a^j, x_{j+1})$ is

$$
Q_{X_{n+2}|X_{n+1}\in I_{i^*}^i}^{\min}(t_a^j, x_{j+1}) = \sum_{l=a}^{s_j} M_{X_{n+2}|X_{n+1}\in I^i}(c_l^j, x_{j+1}) \tag{33}
$$

Similarly, to find the optimal values of $\underline{\beta}^j$ and $\underline{\beta}^{c_l^j}$ that maximise $Q_{X_{n+2}|X_{n+1}\in I_{i^*}^i}(t_a^j, x_{j+1})$, as stated in Equation (32), we need to assign all conditional probability masses in the interval (x_j, x_{j+1}) to the right of t_a^j, that is

$$
\sum_{j^*=a}^{s_j} \beta_{j^*+1}^j = 1 \ , \qquad \sum_{j^*=0}^{a-1} \beta_{j^*+1}^j = 0
$$

and

$$
\sum_{j^*=1}^{a-1} \beta_{j^*}^{c_l^j} = 0 \ , \qquad \sum_{j^*=a}^{s_j-l+1} \beta_{j^*}^{c_l^j} = 1
$$

thus, the maximum value of $Q_{X_{n+2}|X_{n+1}\in I^i_{i*}}(t^j_a, x_{j+1})$ is

$$\begin{aligned}
Q^{\max}_{X_{n+2}|X_{n+1}\in I^i_{i*}}(t^j_a, x_{j+1}) &= M_{X_{n+2}|X_{n+1}\in I^i_{i*}}(x_j, x_{j+1}) + \sum_{l=1}^{a-1} M_{X_{n+2}|X_{n+1}\in I^i_{i*}}(c^j_l, x_{j+1}) \\
&\quad + \sum_{l=a}^{s_j} M_{X_{n+2}|X_{n+1}\in I^i_{i*}}(c^j_l, x_{j+1}) \\
&= M_{X_{n+2}|X_{n+1}\in I^i_{i*}}(x_j, x_{j+1}) + \sum_{l=1}^{s_i} M_{X_{n+2}|X_{n+1}\in I^i_{i*}}(c^j_l, x_{j+1}) \\
&= P_{X_{n+2}|X_{n+1}\in I^i_{i*}}(x_j, x_{j+1}) \qquad (34)
\end{aligned}$$

For convenience, the probabilities in Equations (33) and (34) can also be represented as $Q^{\min}_{X_{n+2}\in I^j_a|X_{n+1}\in I^i_{i*}}$ and $Q^{\max}_{X_{n+2}\in I^j_a|X_{n+1}\in I^i_{i*}}$, respectively.

Consequently, the NPI lower probability for the event $X_{n+2} > t$ given $X_{n+1} > t$, for $t \in [t^j_a, t^j_{a+1})$ with $j = 0, 1, \ldots, u+1$ and $a = 0, 1, \ldots, s_j$, is given by t

$$\begin{aligned}
\underline{P}(X_{n+2} > t|X_{n+1} > t) &= Q^{\min}_{X_{n+2}|X_{n+1}\in I^i_{i*}}(t^j_{a+1}, x_{j+1}) + \sum_{z=j+1}^{u+1} P_{X_{n+2}|X_{n+1}\in I^i_{i*}}(x_z, x_{z+1}) \\
&= \sum_{l=a+1}^{s_i} M_{X_{n+2}|X_{n+1}\in I^i_{i*}}(c^j_l, x_{j+1}) + \sum_{z=j+1}^{u+1} P_{X_{n+2}|X_{n+1}\in I^i_{i*}}(x_z, x_{z+1}) \qquad (35)
\end{aligned}$$

The corresponding NPI upper probability for the event $X_{n+2} > t$ given $X_{n+1} > t$, for $t \in [x_j, x_{j+1})$ with $j = 1, 2, \ldots, u+1$ and $a = 0, 1, \ldots, s_j$, is given by

$$\begin{aligned}
\overline{P}(X_{n+2} > t|X_{n+1} > t) &= Q^{\max}_{X_{n+2}|X_{n+1}\in I^i_{i*}}(t^j_a, x_{j+1}) + \sum_{z=j+1}^{u+1} P_{X_{n+2}|X_{n+1}\in I^i_{i*}}(x_z, x_{z+1}) \\
&= \sum_{z=j}^{u+1} P_{X_{n+2}|X_{n+1}\in I^i_{i*}}(x_z, x_{z+1}) \qquad (36)
\end{aligned}$$

One should note that the α approach, which involves minimizing and maximizing Equation (18), is only used to derive NPI lower and upper probabilities for the event $X_{n+1} > t$. On the other hand, the β approach, which involves minimizing and maximizing Equation (32), is only used to derive NPI conditional lower and upper probabilities for the event $X_{n+2} > t$ given that $X_{n+1} > t$. However, both approaches must be used together in order to derive NPI lower and upper probabilities for the event $X_{n+1} > t$ and $X_{n+2} > t$, which will be explained in Section 5.

Example 4.1. This example aims to demonstrate the assumption of rc-$A_{(n+1)}$ for X_{n+2}, which is based on the rc-$A_{(n)}$ assumption for X_{n+1} [15] as presented in Section 4. Specifically, it illustrates how to calculate the NPI conditional lower and upper probabilities for the event $X_{n+2} > t$ given $X_{n+1} > t$, using the data set from Example 3.1. The data set consists of four observations, including one right-censored observation, as shown in Figure 1.

In Example 3.1, the probability distribution for X_5 was partially specified by five M-function values associated with five intervals generated by the 4 observations, using Definition 3.1 (see Figure 2). In that example, we applied rc-$A_{(4)}$ for X_5 considering where the probability mass is for X_5. Based on this. we apply rc-$A_{(5)}$ for X_6 in this example.

Given that X_5 falls in those five intervals created by the $n = 4$ data observations, i.e., $I^0 = (0, x_1), I_1^1 = (x_1, c_1^1), I_2^1 = (c_1^1, x_2), I^2 = (x_2, x_3)$ and $I^3 = (x_3, \infty)$, respectively, so there are five cases of which X_5 falls into these intervals. Then, we consider X_6 depending on X_5 being in a specific interval. This enables the probability distribution for X_6 to be partially specified by conditional M-function values assigned to six intervals formed by the 5 observations including X_5, using Definition 4.1 separately for each case.

As a result of applying Definition 4.1 with the assumption rc-$A_{(5)}$ given by Equations (27) and (28), these conditional M-function values for X_6 given $X_5 \in \{I^0, I_1^1, I_2^1, I^2, I^3\}$, can be obtained as follows (see Figure 3).

Case 1: Given $X_5 \in I^0 = (0, x_1)$, the conditional M-function values for $X_6|X_5 \in I^0$, using Definition 4.1 with the assumption rc-$A_{(5)}$ given by Equations (27) and (28), are shown in the first box of Figure 3.

Since there is no censored observation in intervals $(0, x_5), (x_5, x_1)$, (x_2, x_3) and (x_3, ∞), respectively, the corresponding values β_1^0, β_1^1, β_1^3 and β_1^4 introduced to these intervals are equal to one, i.e. $\beta_1^0 = \beta_1^1 = \beta_1^3 = \beta_1^4 = 1$, as discussed in Section 4. For $c_1^2 \in (x_1, x_2)$ given $X_5 \in (0, x_1)$, the conditional M-function value $\frac{1}{6}$ that is assigned to interval (x_1, x_2) will be split up, based on Definition 4.1, and assigned to two sub-intervals with the M-function value $\frac{1}{6}\beta_1^2$ assigned to the sub-interval (x_1, c_1^2) and the M-function value $\frac{1}{6}\beta_2^2$ assigned to the sub-interval (c_1^2, x_2), where both β_1^2 and β_2^2 take values between 0 and 1, and $\beta_1^2 + \beta_2^2 = 1$. Also, based on Definition 4.1 and Equation (28), the M-function value $\frac{1}{18}\beta_1^{c_1^2}$ is assigned to the sub-interval (c_1^2, x_2), where $\beta_1^{c_1^2} = 1$. Thus, the conditional M-function values for the event $X_6|X_5 \in I^0$ are

$$M_{X_6|X_5 \in I^0}(0, x_5) = \frac{1}{6} \tag{37}$$

$$M_{X_6|X_5 \in I^0}(x_5, x_1) = \frac{1}{6} \tag{38}$$

$$\begin{aligned} M_{X_6|X_5 \in I^0}(x_1, c_1^2) &= \frac{1}{6}\beta_1^2 \\ M_{X_6|X_5 \in I^0}(c_1^2, x_2) &= \frac{1}{6}\beta_2^2 + \frac{1}{18}\beta_1^{c_1^2} \\ M_{X_6|X_5 \in I^0}(x_2, x_3) &= \frac{1}{6} + \frac{1}{18} \\ M_{X_6|X_5 \in I^0}(x_3, \infty) &= \frac{1}{6} + \frac{1}{18} \end{aligned} \tag{39}$$

where the total conditional probability mass for $X_6 \in (0, x_1)$ given $X_5 \in (0, x_1)$, given in Equations (37) and (38), is 1/6+1/6=2/6, see *Case 1* in the first box of Figure 3.

Based on these conditional M-function values, we can derive the conditional probability for the event $X_6 \in (x_1, x_2)$ given $X_5 \in (0, x_1)$, by summing the probability masses assigned to the sub-intervals (x_1, c_1^2) and (c_1^2, x_2), so $P_{X_6|X_5 \in I^0}(x_1, x_2) = \frac{1}{6}\beta_1^2 + \frac{1}{6}\beta_2^2 + \frac{1}{18}\beta_1^{c_1^2} = \frac{1}{6}(\beta_1^2 + \beta_2^2) + \frac{1}{18}\beta_1^{c_1^2}$, and as discussed in Section 4, $\beta_1^2 + \beta_2^2 = 1$ and $\beta_1^{c_1^2} = 1$, so $P_{X_6|X_5 \in I^0}(x_1, x_2) =$

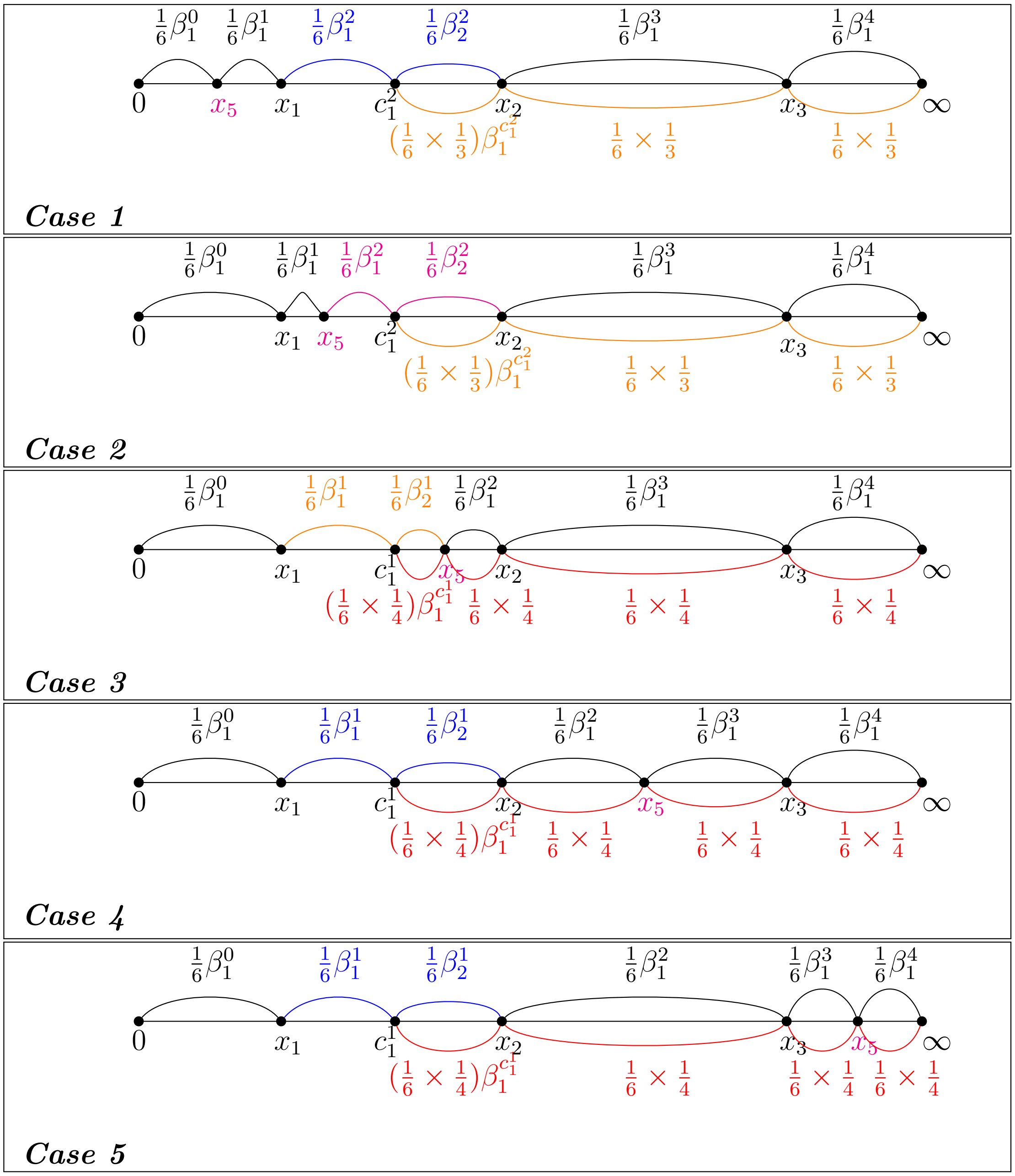

Figure 3: The conditional probabilities for $X_6|X_5$, Example 5.1

$\frac{1}{6} + \frac{1}{18} = \frac{4}{18}$. Moreover, the conditional probabilities for X_6 to be in intervals (x_2, x_3) or (x_3, ∞), given $X_5 \in (0, x_1)$, are $P_{X_6|X_5 \in I^0}(x_2, x_3) = P_{X_6|X_5 \in I^0}(x_3, \infty) = \frac{1}{6} + \frac{1}{18} = \frac{4}{18}$.

Then from *Case 1* in which $X_5 \in (0, x_1)$, we now consider the event $X_6 > t$ given $X_5 > t$, where $t \in (0, x_1)$ (see the first box of Figure 3). By assigning all conditional probability masses that must be within (t, ∞), Equation (35) is used to determine the NPI lower conditional probability for the event $X_6 > t$ given $X_5 > t$, where $t \in (0, x_1)$. Thus

$$\begin{aligned}
\underline{P}(X_6 > t | X_5 > t) &= Q^{\min}_{X_6|X_5 \in I^0}(0, x_1) + \sum_{z=2}^{4} P_{X_6|X_5 \in I^0}(x_z, x_{z+1}) \\
&= M_{X_6|X_5 \in I^0}(x_5, x_1) + P_{X_6|X_5 \in I^0}(x_1, x_2) + P_{X_6|X_5 \in I^0}(x_2, x_3) \\
&+ P_{X_6|X_5 \in I^0}(x_3, \infty) \\
&= \frac{1}{6} + \frac{4}{18} + \frac{4}{18} + \frac{4}{18} = \frac{5}{6}
\end{aligned}$$

where the value of $Q^{\min}_{X_6|X_5 \in I^0}(0, x_1)$ is obtained by using Equation (33).

The NPI upper conditional probability for the event $X_6 > t$ given $X_5 > t$, where $t \in (0, x_1)$, is derived by assigning all conditional probability masses that could be within (t, ∞) using Equation (36). Thus

$$\begin{aligned}
\overline{P}(X_6 > t | X_5 > t) &= Q^{\max}_{X_6|X_5 \in I^0}(0, x_1) + \sum_{z=2}^{4} P_{X_6|X_5 \in I^0}(x_j, x_{j+1}) \\
&= P_{X_6|X_5 \in I^0}(0, x_1) + P_{X_6|X_5 \in I^0}(x_1, x_2) + P_{X_6|X_5 \in I^0}(x_2, x_3) \\
&+ P_{X_6|X_5 \in I^0}(x_3, \infty) \\
&= \frac{2}{6} + \frac{4}{18} + \frac{4}{18} + \frac{4}{18} = 1
\end{aligned}$$

where the value of $Q^{\max}_{X_6|X_5 \in I^0}(0, x_1)$ is obtained by using Equation (34).

Case 2: Given $X_5 \in I_1^1 = (x_1, c_1^1)$, the conditional M-function values for $X_6 | X_5 \in I_0^1$, using Definition 4.1 with the assumption rc-$A_{(5)}$ given by Equations (27) and (28), are shown in the second box of Figure 3. Due to the fact that no censoring is involved in intervals $(0, x_1), (x_1, x_5)$, (x_2, x_3) and (x_3, ∞), respectively, the values β_1^0, β_1^1, β_1^3 and β_1^4 corresponding to these intervals are equal to 1, as stated in Section 4.

By using Equation (27), based on the assumption rc-$A_{(5)}$, the conditional M-function value for $X_6 \in (x_1, x_5) | X_5 \in I_1^1$ is $\frac{1}{6}$. For $c_1^2 \in (x_5, x_2)$ given $X_5 \in I_1^1$, the conditional M-function value $\frac{1}{6}$ that is assigned to interval (x_5, x_2) will be split up and assigned to two sub-intervals with the M-function value $\frac{1}{6}\beta_1^2$ assigned to the sub-interval (x_5, c_1^2) as well as the M-function value $\frac{1}{6}\beta_2^2$ assigned to the sub-interval (c_1^2, x_2), where both β_1^2 and β_2^2 take values between 0 and 1, and $\beta_1^2 + \beta_2^2 = 1$. Also, based on Definition 4.1 and Equation (28), the M-function value $\frac{1}{18}\beta_1^{c_1^2}$ is assigned to the sub-interval (c_1^2, x_2), where $\beta_1^{c_1^2} = 1$.

Thus, the conditional M-function values for the event $X_6|X_5 \in I_1^1$ are

$$M_{X_6|X_5\in I_1^1}(0,x_1) = \frac{1}{6}$$
$$M_{X_6|X_5\in I_1^1}(x_1,x_5) = \frac{1}{6} \tag{40}$$
$$M_{X_6|X_5\in I_1^1}(x_5,c_1^2) = \frac{1}{6}\beta_1^2 \tag{41}$$
$$M_{X_6|X_5\in I_1^1}(c_1^2,x_2) = \frac{1}{6}\beta_2^2 + \frac{1}{18}\beta_1^{c_1^2}$$
$$M_{X_6|X_5\in I_1^1}(x_2,x_3) = \frac{1}{6} + \frac{1}{18}$$
$$M_{X_6|X_5\in I_1^1}(x_3,\infty) = \frac{1}{6} + \frac{1}{18}$$

where the total conditional probability mass for $X_6 \in (x_1, c_1^1)$ given $X_5 \in (x_1, c_1^1)$, given in Equations (40) and (41), is 1/6 (1+β_1^2), where $\beta_1^2 \in [0,1]$, see *Case 2* in the second box of Figure 3.

From *Case 2*, where $X_5 \in I_1^1 = (x_1, c_1^1)$, we use Equation (35) to derive the NPI conditional lower probability for the event $X_6 > t$ given $X_5 > t$, where $t \in (x_1, c_1^1)$, as

$$\begin{aligned}\underline{P}(X_6 > t|X_5 > t) &= Q^{\min}_{X_6|X_5\in I_1^1}(x_1,x_2) + \sum_{z=3}^{4} P_{X_6|X_5\in I_1^1}(x_z,x_{z+1}) \\ &= M_{X_6|X_5\in I_1^1}(x_5,x_2) + P_{X_6|X_5\in I_1^1}(x_2,x_3) + P_{X_6|X_5\in I_1^1}(x_3,\infty) \\ &= \frac{4}{18} + \frac{4}{18} + \frac{4}{18} = \frac{2}{3}\end{aligned}$$

where the value of $Q^{\min}_{X_6|X_5\in I_1^1}(x_1,x_2)$ is obtained by using Equation (33), i.e., $Q^{\min}_{X_6|X_5\in I_1^1}(x_1,x_2) = \frac{1}{6}\beta_1^2 + \frac{1}{6}\beta_2^2 + \frac{1}{18}\beta_1^{c_1^2} = \frac{1}{6}(\beta_1^2+\beta_2^2) + \frac{1}{18}\beta_1^{c_1^2}$. And for $\beta_1^2+\beta_2^2 = 1$ and $\beta_1^{c_1^2} = 1$, $Q^{\min}_{X_6|X_5\in I_1^1}(x_1,x_2) = \frac{1}{6} + \frac{1}{18} = \frac{4}{18}$.

The NPI upper conditional probability for the event $X_6 > t$ given $X_5 > t$, where $t \in (x_1, c_1^1)$, is derived by using Equation (36) as follows,

$$\begin{aligned}\overline{P}(X_6 > t|X_5 > t) &= Q^{\max}_{X_6|X_5\in I_1^1}(x_1,x_2) + \sum_{z=3}^{4} P_{X_6|X_5\in I_1^1}(x_z,x_{z+1}) \\ &= P_{X_6|X_5\in I_1^1}(x_1,x_2) + P_{X_6|X_5\in I_1^1}(x_2,x_3) + P_{X_6|X_5\in I_1^1}(x_3,\infty) \\ &= \frac{7}{18} + \frac{4}{18} + \frac{4}{18} = \frac{5}{6}\end{aligned}$$

where the value of $Q^{\max}_{X_6|X_5\in I_1^1}(x_1,x_2)$ is obtained by using Equation (34), i.e., $Q^{\max}_{X_6|X_5\in I_1^1}(x_1,x_2) = P_{X_6|X_5\in I_1^1}(x_1,x_5) + P_{X_6|X_5\in I_1^1}(x_5,x_2) = \frac{1}{6} + \frac{4}{18} = \frac{7}{18}$.

Case 3: Given $X_5 \in I_2^1 = (c_1^1, x_2)$, the conditional M-function values for $X_6|X_5 \in I_2^1$, using Definition 4.1 with the assumption rc-$A_{(5)}$ given by Equations (27) and (28), are shown in the third box of Figure 3.

The β_1^0, β_1^2, β_1^3 and β_1^4 values corresponding to the intervals $(0,x_1)$, (x_5,x_2), (x_2,x_3) and (x_3,∞), respectively, are equal to 1, since there no censoring is involved in these intervals.

For $c_1^1 \in (x_1, x_5)$ given $X_5 \in I_2^1$, the conditional M-function value $\frac{1}{6}$ that is assigned to interval (x_1, x_5) will be split up and assigned to two sub-intervals with the M-function value $\frac{1}{6}\beta_1^1$ assigned to the sub-interval (x_1, c_1^1) as well as the M-function value $\frac{1}{6}\beta_2^1$ assigned to the sub-interval (c_1^1, x_5), where both β_1^1 and β_2^1 take values between 0 and 1, and $\beta_1^1 + \beta_2^1 = 1$. Also, based on Definition 4.1 and Equation (28), the M-function value $\frac{1}{24}\beta^{c_1^1}$ is assigned to the sub-interval (c_1^1, x_5), where $\beta^{c_1^1} = 1$. Thus, the conditional M-function values for the event $X_6 | X_5 \in I_2^1$ are

$$
\begin{aligned}
M_{X_6|X_5\in I_2^1}(0, x_1) &= \frac{1}{6} \\
M_{X_6|X_5\in I_2^1}(x_1, c_1^1) &= \frac{1}{6}\beta_1^1 \\
M_{X_6|X_5\in I_2^1}(c_1^1, x_5) &= \frac{1}{6}\beta_2^1 + \frac{1}{24}\beta_1^{c_1^1} \qquad (42)\\
M_{X_6|X_5\in I_2^1}(x_5, x_2) &= \frac{1}{6} + \frac{1}{24} \qquad (43)\\
M_{X_6|X_5\in I_2^1}(x_2, x_3) &= \frac{1}{6} + \frac{1}{24} \\
M_{X_6|X_5\in I_2^1}(x_3, \infty) &= \frac{1}{6} + \frac{1}{24}
\end{aligned}
$$

where the total conditional probability mass for $X_6 \in (c_1^1, x_2)$ given $X_5 \in (c_1^1, x_2)$, given in Equations (42) and (43), is 1/6 (β_2^1 +1/4 $\beta_1^{c_1^1}$ + 5/4), where $\beta_2^1 \in [0, 1]$ and $\beta_1^{c_1^1} = 1$, see *Case 3* in the third box of Figure 3.

From *Case 3*, where $X_5 \in I_2^1 = (c_1^1, x_2)$, we use Equation (35) to derive the NPI lower conditional probability for the event $X_6 > t$ given $X_5 > t$, where $t \in (c_1^1, x_2)$, as

$$
\begin{aligned}
\underline{P}(X_6 > t | X_5 > t) &= Q^{\min}_{X_6|X_5\in I_2^1}(x_1, x_2) + \sum_{z=3}^{4} P_{X_6|X_5\in I_2^1}(x_z, x_{z+1}) \\
&= M_{X_6|X_5\in I_2^1}(x_5, x_2) + P_{X_6|X_5\in I_2^1}(x_2, x_3) + P_{X_6|X_5\in I_2^1}(x_3, \infty) \\
&= \frac{5}{24} + \frac{5}{24} + \frac{5}{24} = \frac{5}{8}
\end{aligned}
$$

where the value of $Q^{\min}_{X_6|X_5\in I_2^1}(x_1, x_2)$ is obtained by using Equation (33), i.e., $Q^{\min}_{X_6|X_5\in I_2^1}(x_1, x_2) = \frac{1}{6} + \frac{1}{24} = \frac{5}{24}$.

The NPI upper conditional probability for the event $X_6 > t$ given $X_5 > t$, where $t \in (c_1^1, x_2)$, is derived by using Equation (36) as follows,

$$
\begin{aligned}
\overline{P}(X_6 > t | X_5 > t) &= Q^{\max}_{X_6|X_5\in I_2^1}(x_1, x_2) + \sum_{z=3}^{4} P_{X_6|X_5\in I_2^1}(x_z, x_{z+1}) \\
&= P_{X_6|X_5\in I_2^1}(x_1, x_2) + P_{X_6|X_5\in I_2^1}(x_2, x_3) + P_{X_6|X_5\in I_2^1}(x_3, \infty) \\
&= \frac{10}{24} + \frac{5}{24} + \frac{5}{24} = \frac{5}{6}
\end{aligned}
$$

where the value of $Q^{\max}_{X_6|X_5\in I_2^1}(x_1, x_2)$ is obtained by using Equation (34), i.e., $Q^{\max}_{X_6|X_5\in I_2^1}(x_1, x_2) = P_{X_6|X_5\in I_2^1}(x_1, x_5) + P_{X_6|X_5\in I_2^1}(x_5, x_2) = \frac{5}{24} + \frac{5}{24} = \frac{10}{24}$.

Case 4: Given $X_5 \in I^2 = (x_2, x_3)$, the conditional M-function values for $X_6|X_5 \in I^2$, using Definition 4.1 with the assumption rc-$A_{(5)}$ given by Equations (27) and (28), are shown in the fourth box of Figure 3.

The β_1^0, β_1^2, β_1^3 and β_1^4 values corresponding to the intervals $(0, x_1), (x_2, x_5), (x_5, x_3)$ and (x_3, ∞), respectively, are equal to 1, since there no censoring is involved in these intervals. For $c_1^1 \in (x_1, x_2)$ given $X_5 \in I^2$, the conditional M-function value $\frac{1}{6}$ that is assigned to interval (x_1, x_2) will be split up and assigned to two sub-intervals with the M-function value $\frac{1}{6}\beta_1^1$ assigned to the sub-interval (x_1, c_1^1) as well as the M-function value $\frac{1}{6}\beta_2^1$ assigned to the sub-interval (c_1^1, x_2), where both β_1^1 and β_2^1 take values between 0 and 1, and $\beta_1^1 + \beta_2^1 = 1$. Also, based on Definition 4.1 and Equation (28), the M-function value $\frac{1}{24}\beta^{c_1^1}$ is assigned to the sub-interval (c_1^1, x_2), where $\beta^{c_1^1} = 1$. Thus, the conditional M-function values for the event $X_6|X_5 \in I^2$ are

$$\begin{aligned}
M_{X_6|X_5 \in I^2}(0, x_1) &= \frac{1}{6} \\
M_{X_6|X_5 \in I^2}(x_1, c_1^1) &= \frac{1}{6}\beta_1^1 \\
M_{X_6|X_5 \in I^2}(c_1^1, x_2) &= \frac{1}{6}\beta_2^1 + \frac{1}{24}\beta_1^{c_1^1} \\
M_{X_6|X_5 \in I^2}(x_2, x_5) &= \frac{1}{6} + \frac{1}{24} \qquad (44) \\
M_{X_6 X_5 \in I^2}(x_5, x_3) &= \frac{1}{6} + \frac{1}{24} \qquad (45) \\
M_{X_6|X_5 \in I^2}(x_3, \infty) &= \frac{1}{6} + \frac{1}{24}
\end{aligned}$$

where the total conditional probability mass for $X_6 \in (x_2, x_3)$ given $X_5 \in (x_2, x_3)$, given in Equations (44) and (45), is 5/24+5/24=10/24, see *Case 4* in the fourth box of Figure 3.

From *Case 4*, where $X_5 \in I^2 = (x_2, x_3)$, we use Equation (35) to derive the NPI lower conditional probability for the event $X_6 > t$ given $X_5 > t$, where $t \in (x_2, x_3)$, as

$$\begin{aligned}
\underline{P}(X_6 > t|X_5 > t) &= Q^{\min}_{X_6|X_5 \in I^2}(x_2, x_3) + \sum_{z=4}^{4} P_{X_6|X_5 \in I^2}(x_z, x_{z+1}) \\
&= M_{X_6|X_5 \in I^2}(x_5, x_3) + P_{X_6|X_5 \in I^2}(x_3, \infty) \\
&= \frac{5}{24} + \frac{5}{24} = \frac{5}{12}
\end{aligned}$$

where the value of $Q^{\min}_{X_6|X_5 \in I^2}(x_2, x_3)$ is obtained by using Equation (33), i.e., $Q^{\min}_{X_6|X_5 \in I^2}(x_2, x_3) = \frac{1}{6} + \frac{1}{24} = \frac{5}{24}$.

The NPI upper conditional probability for the event $X_6 > t$ given $X_5 > t$, where $t \in (x_2, x_3)$, is derived by using Equation (36), as follows,

$$\begin{aligned}
\overline{P}(X_6 > t|X_5 > t) &= Q^{\max}_{X_6|X_5 \in I^2}(x_2, x_3) + \sum_{z=4}^{4} P_{X_6|X_5 \in I^2}(x_z, x_{z+1}) \\
&= P_{X_6|X_5 \in I^2}(x_2, x_5) + P_{X_6|X_5 \in I^2}(x_5, x_3) + P_{X_6|X_5 \in I^2}(x_3, \infty) \\
&= \frac{5}{24} + \frac{5}{24} + \frac{5}{24} = \frac{5}{8}
\end{aligned}$$

$t \in (.)$	$\underline{P}(X_6 > t \mid X_5 > t)$	$\overline{P}(X_6 > t \mid X_5 > t)$
$(0, x_1)$	$\frac{5}{6}$	1
(x_1, c_1^1)	$\frac{2}{3}$	$\frac{5}{6}$
(c_1^1, x_2)	$\frac{5}{8}$	$\frac{5}{6}$
(x_2, x_3)	$\frac{5}{12}$	$\frac{5}{8}$
(x_3, ∞)	$\frac{5}{24}$	$\frac{5}{12}$

Table 2: Lower and upper conditional probabilities for the event $(X_6 > t \mid X_5 > t)$, Example 4.1.

where the value of $Q^{\max}_{X_6|X_5 \in I^2}(x_2, x_3)$ is obtained by using Equation (34), i.e., $Q^{\max}_{X_6|X_5 \in I^2}(x_2, x_3) = P_{X_6|X_5 \in I^2}(x_2, x_5) + P_{X_6|X_5 \in I^2}(x_5, x_3) = \frac{5}{24} + \frac{5}{24} = \frac{10}{24}$.

Case 5: Given $X_5 \in I^3 = (x_3, \infty)$, the conditional M-function values for $X_6|X_5 \in I^3$, using Definition 4.1 with the assumption rc-$A_{(4+1)}$ given by Equations (27) and (28), are shown in the fifth box of Figure 3.

The β_1^0, β_1^2, β_1^3 and β_1^4 values corresponding to the intervals $(0, x_1), (x_2, x_3)$, (x_3, x_5) and (x_5, ∞), respectively, are equal to 1, since there no censoring is involved in these intervals. For $c_1^1 \in (x_1, x_2)$ given $X_5 \in I^3$, the conditional M-function value $\frac{1}{6}$ that is assigned to interval (x_1, x_2) will be split up and assigned to two sub-intervals with the M-function value $\frac{1}{6}\beta_1^1$ assigned to the sub-interval (x_1, c_1^1) as well as the M-function value $\frac{1}{6}\beta_2^1$ assigned to the sub-interval (c_1^1, x_2), where both β_1^1 and β_2^1 take values between 0 and 1, and $\beta_1^1 + \beta_2^1 = 1$. Also, based on Definition 4.1 and Equation (28), the M-function value for X_6, $\frac{1}{24}\beta^{c_1^1}$, is assigned to the sub-interval (c_1^1, x_2), where $\beta^{c_1^1} = 1$. Thus, the conditional M-function values for the event $X_6|X_5 \in I^3$ are

$$
\begin{aligned}
M_{X_6|X_5 \in I^3}(0, x_1) &= \frac{1}{6} \\
M_{X_6|X_5 \in I^3}(x_1, c_1^1) &= \frac{1}{6}\beta_1^1 \\
M_{X_6|X_5 \in I^3}(c_1^1, x_2) &= \frac{1}{6}\beta_2^1 + \frac{1}{24}\beta_1^{c_1^1} \\
M_{X_6|X_5 \in I^3}(x_2, x_3) &= \frac{1}{6} + \frac{1}{24} \\
M_{X_6|X_5 \in I^3}(x_3, x_5) &= \frac{1}{6} + \frac{1}{24} \qquad (46) \\
M_{X_6|X_5 \in I^3}(x_5, \infty) &= \frac{1}{6} + \frac{1}{24} \qquad (47)
\end{aligned}
$$

where the total conditional probability mass for $X_6 \in (x_3, \infty)$ given $X_5 \in (x_3, \infty)$, given in Equations (46) and (47), is 5/24+5/24=10/24, see *Case 5* in the fifth box of Figure 3.

From *Case 5*, where $X_5 \in I^3 = (x_3, \infty)$, we use Equation (35) to derive the NPI lower conditional probability for the event $X_6 > t$ given $X_5 > t$, where $t \in (x_3, \infty)$, as

$$
\underline{P}(X_6 > t \mid X_5 > t) = Q^{\min}_{X_6|X_5 \in I^3}(x_3, \infty) = M_{X_6|X_5 \in I^3}(x_5, \infty) = \frac{5}{24}
$$

where the value of $Q^{\min}_{X_6|X_5 \in I^2}(x_2, x_3)$ is obtained by using Equation (33).

The NPI upper conditional probability for the event $X_6 > t$ given $X_5 > t$, where $t \in (x_3, \infty)$, is derived by using Equation (36), as follows.

$$\overline{P}(X_6 > t | X_5 > t) = Q^{\max}_{X_6|X_5 \in I^3}(x_3, \infty) = P_{X_6|X_5 \in I^3}(x_3, x_5) + P_{X_6|X_5 \in I^3}(x_5, \infty)$$
$$= \frac{5}{24} + \frac{5}{24} = \frac{5}{12}$$

where the value of $Q^{\max}_{X_6|X_5 \in I^3}(x_2, x_3)$ is obtained by using Equation (34). Therefore, the NPI lower and upper conditional probabilities for the event that $X_6 > t$ given $X_5 > t$, are given in Table 2. The values of the NPI lower and upper probabilities at observations are easily derived from Table 2, using the fact that the lower probability is continuous from the left at all observations, given by Equation (35), and the upper probability is continuous from the right at event times, given by Equation (36). An effect of conditioning on the second future observation X_5 to be in the final interval (x_3, ∞), the NPI lower probability for $X_6 \in (x_3, \infty)$ is positive which is given by the M-function value $\frac{5}{24}$ that assigned to the sub-interval (x_5, ∞).

In the next section, we present NPI lower and upper probabilities for the event $X_{n+1} > t$ and $X_{n+2} > t$, based on the results presented in Sections 3 and 4.

5. Lower and upper probabilities for $X_{n+1} > t$, $X_{n+2} > t$

This section derives the NPI lower and upper probabilities for the event that both future observations X_{n+1} and X_{n+2} are greater than time $t > 0$. The notation used in this section follow those introduced in Sections 3 and 4. Let $I^i_{i^*} = (t^i_{i^*}, t^i_{i^*+1})$ be an interval created by the n data observations, $i = 0, 1, 2, \ldots, u$ and $i^* = 1, 2, \ldots, s_i$, that is we have $n + 1$ intervals created by the data, and let $I^i = (x_i, x_{i+1})$ be the ith interval created by two consecutive failures and $I^i_a = (t^i_a, x_{i+1})$, $i = 0, 1, \ldots, u$ and $a = 0, 1, \ldots, s_i$. Furthermore, let $M_{X_{n+1} \in I^j_{j^*}}$ be the M-function values for X_{n+1}, based on the assumption rc-$A_{(n)}$ [15], as defined in Definition 3.1, where $j = 0, 1, \ldots, u$ and $j^* = 1, 2, \ldots, s_j$. Let $P_{X_{n+1} \in I^j}$ be the probabilities for X_{n+1} to belong to the intervals $I^j = (x_j, x_{j+1})$ as given by Equation (15). Let $M_{X_{n+2} \in I^k_{k^*} | X_{n+1} \in I^j_{j^*}}$ be the conditional M-function values for $X_{n+2} \in I^k = (x_k, x_{k+1})$, $k = 0, 1, \ldots, u$, $k^* = 1, 2, \ldots, s_k$, based on the assumption rc-$A_{(n+1)}$, as defined in Definition 4.1. Let $P_{X_{n+2} \in I^k | X_{n+1} \in I^j}$ be the conditional probabilities for the event $\{X_{n+2} \in I^k | X_{n+1} \in I^j\}$, as given by Equation (29).

To find the NPI lower and upper probabilities for the joint event where $X_{n+1} > t$ and $X_{n+2} > t$ for all $t > 0$, we will utilize the results from Sections 3 and 4. Firstly, we will derive the NPI upper probability for the event where $X_{n+1} > t$ and $X_{n+2} > t$ for $t \in [x_i, x_{i+1})$, where $i = 0, 1, \ldots, u$.

Theorem 5.1. The NPI upper probability is derived by summing all probability masses that can be to the right of t. This means all M-function values assigned to intervals $I^k_{k^*}, I^j_{j^*} \in \{I^i_a, \ldots, I^i_{s_i}, I^{i+1}_0, \ldots, I^{i+1}_{s_{i+1}}, \ldots, I^u_{s_u}\}$ will lead to the following NPI upper probability

$$\overline{P}(X_{n+1} > t, X_{n+2} > t) = \sum_{j=i}^{u} \sum_{k=i}^{u} P_{X_{n+2} \in I^k | X_{n+1} \in I^j} P_{X_{n+1} \in I^j} \tag{48}$$

Proof. There are four terms of summations that, when added together, lead to derive the NPI upper probability for the event $X_{n+1} > t$ and $X_{n+2} > t$. We refer to these terms as J_1, J_2, J_3, and J_4, stated in Equations (49), (50), (51), and (52), respectively, which are illustrated in detail below.

First, we sum over $I^k_{k^*} \in \{I^{i+1}_0, \ldots, I^{i+1}_{s_{i+1}}, \ldots, I^u_{s_u}\}$ and $I^j_{j^*} \in \{I^{i+1}_0, \ldots, I^{i+1}_{s_{i+1}}, \ldots, I^u_{s_u}\}$, which is equivalent to summing over the intervals I^k and I^j for $k, j \in \{i+1, \ldots, u\}$. This will lead to constant probabilities using Equations (15) and (29), respectively, so these probabilities are not functions of the α's or β's, so no optimisation is required here. We can write these summations terms as

$$\begin{aligned}
J_1 &= \sum_{j=i+1}^{u} \sum_{k=i+1}^{u} \sum_{j^*=0}^{s_j} \sum_{k^*=0}^{s_k} M_{X_{n+2} \in I^k_{k^*} | X_{n+1} \in I^j_{j^*}} M_{X_{n+1} \in I^j_{j^*}} \\
&= \sum_{j=i+1}^{u} \sum_{k=i+1}^{u} \left[\sum_{j^*=0}^{s_j} M_{X_{n+1} \in I^j_{j^*}} \right] \left[\sum_{k^*=0}^{s_k} M_{X_{n+2} \in I^k_{k^*} | X_{n+1} \in I^j_{j^*}} \right] \\
&= \sum_{j=i+1}^{u} \sum_{k=i+1}^{u} P_{X_{n+2} \in I^k | X_{n+1} \in I^j} P_{X_{n+1} \in I^j} \qquad (49)
\end{aligned}$$

where summing all M-function values for $X_{n+1} \in I^j_{j^*}$ as well as summing up all conditional M-function values for $X_{n+2} \in I^k_{k^*} | X_{n+1} \in I^j_{j^*}$, in the second equality, lead to the probabilities for the event $X_{n+1} \in I^j$, as well as to the conditional probability masses for the event $X_{n+1} \in I^j | X_{n+1} \in I^j$, for $j = i+1, \ldots, u$ and $k = i+1, \ldots, u$, as in the third equality. Thus, we have advanced from the second equality to the third equality by using Equations (15) and (29), respectively.

Secondly, we sum over $I^k_{k^*} \in \{I^i_a, \ldots, I^i_{s_i}\}$ and $I^j_{j^*} \in \{I^{i+1}_0, \ldots, I^{i+1}_{s_{i+1}}, \ldots, I^u_{s_u}\}$, which will lead to a function of the β's only, so we need to maximise this function. This leads to

$$\begin{aligned}
J_2 &= \sum_{j=i+1}^{u} \sum_{j^*=0}^{s_j} \sum_{k^*=a}^{s_i} M_{X_{n+2} \in I^k_{k^*} | X_{n+1} \in I^j_{j^*}} M_{X_{n+1} \in I^j_{j^*}} \\
&= \sum_{j=i+1}^{u} \sum_{j^*=0}^{s_j} M_{X_{n+1} \in I^j_{j^*}} \sum_{k^*=a}^{s_i} M_{X_{n+2} \in I^k_{k^*} | X_{n+1} \in I^j_{j^*}} \\
&= \sum_{j=i+1}^{u} Q^{\max}_{X_{n+2} \in I^i_a | X_{n+1} \in I^j} P_{X_{n+1} \in I^j} \\
&= \sum_{j=i+1}^{u} P_{X_{n+2} \in I^i | X_{n+1} \in I^j} P_{X_{n+1} \in I^j} \qquad (50)
\end{aligned}$$

where in the third equality, the function $Q^{\max}_{X_{n+2} \in I^i_a | X_{n+1} \in I^j_{j^*}}$ is considered to maximise the conditional probability mass for $X_{n+2} \in I^i_a = (t^i_a, x_{i+1})$ given $X_{n+1} \in I^j_{j^*}$, where $j = i + 1, \ldots, u$, by assigning all conditional M-function values within the interval $I^i = (x_i, x_{i+1})$ to the right of t^i_a. This leads to the conditional probability mass for the event $X_{n+2} \in I^i$ given $X_{n+1} \in I^j$, where $I^i = (x_i, x_{i+1})$ and $j = i + 1, \ldots, u$. Then, we are able to move from the third equality to the fourth equality via the product of the conditional probability $P_{X_{n+2}\in I^i|X_{n+1}\in I^j}$ and the probability mass for the event that $X_{n+1} \in I^j$, where $I^i = (x_i, x_{i+1})$, $I^j = (x_j, x_{j+1})$ and $j = i+1, \ldots, u$. The function $Q^{\max}_{X_{n+2}\in I^i_a|X_{n+1}\in I^j_{j^*}}$, which is a function of the β's only, is maximised by using Equation (34).

Thirdly, we sum over $I^k_{k^*} \in \{I^{i+1}_0, \ldots, I^{i+1}_{s_{i+1}}, \ldots, I^u_{s_u}\}$ and $I^j_{j^*} \in \{I^i_a, \ldots, I^i_{s_i}\}$, which will lead to a function of the α's only, so we need to maximise this function. This leads to

$$\begin{aligned}
J_3 &= \sum_{k=i+1}^{u} \sum_{j^*=a}^{s_i} \sum_{k^*=0}^{s_i} M_{X_{n+2}\in I^k_{k^*}|X_{n+1}\in I^i_{j^*}} M_{X_{n+1}\in I^i_{j^*}} \\
&= \sum_{k=i+1}^{u} \sum_{j^*=a}^{s_i} P_{X_{n+2}\in I^k|X_{n+1}\in I^i_{j^*}} M_{X_{n+1}\in I^i_{j^*}} \\
&= \sum_{k=i+1}^{u} P_{X_{n+2}\in I^k|X_{n+1}\in I^i_{j^*}} Q^{\max}_{X_{n+1}\in I^i_a} \\
&= \sum_{k=i+1}^{u} P_{X_{n+2}\in I^k|X_{n+1}\in I^i} P_{X_{n+1}\in I^i} \qquad (51)
\end{aligned}$$

where in the third equality, the function $Q^{\max}_{X_{n+1}\in I^i_a}$ is considered to maximise the probability mass for $X_{n+1} \in I^j_a = (t^j_a, x_{j+1})$, by assigning all M-function values within the interval $I^j = (x_j, x_{j+1})$ to the right of t^i_a, using Equation (20). This leads to the probability mass for the event $X_{n+1} \in I^i = (x_i, x_{i+1})$. This has advanced from the third equality to the fourth equality through the product of the conditional probabilities for the event that $X_{n+2} \in I^k|X_{n+1} \in I^i$, and the probability mass for the event that $X_{n+1} \in I^i$, where $k = i+1, \ldots, u$.

Finally, we sum over $I^k_{k^*} \in \{I^i_a, \ldots, I^i_{s_i}\}$ and $I^j_{j^*} \in \{I^i_a, \ldots, I^i_{s_i}\}$, which will lead to functions of the α's and β's, so we need to maximise both functions. This leads to

$$\begin{aligned}
J_4 &= \sum_{j^*=a}^{s_i} \sum_{k^*=a}^{s_i} M_{X_{n+2}\in I^i_{k^*}|X_{n+1}\in I^i_{j^*}} M_(X_{n+1} \in I^i_{j^*}) \\
&= Q^{\max}_{X_{n+2}\in I^i_a|X_{n+1}\in I^i_{j^*}} Q^{\max}_{X_{n+1}\in I^i_a} \\
&= P_{X_{n+2}\in I^i|X_{n+1}\in I^i} P_{X_{n+1}\in I^i} \qquad (52)
\end{aligned}$$

where in the second equality, the function $Q^{\max}_{X_{n+1}\in I^i_a}$ is considered to maximise the probability mass for $X_{n+1} \in I^i_a = (t^i_a, x_{i+1})$, by assigning all M-function values within the interval $I^j = (x_j, x_{j+1})$ to the right of t^i_a, using Equation (20), which leads to the probability mass $P_{X_{n+1}\in I^i}$. Also, the function $Q^{\max}_{X_{n+2}\in I^i_a|X_{n+1}\in I^i_{j^*}}$ is considered to maximise the conditional probability mass for $X_{n+2} \in I^i_a = (t^i_a, x_{i+1})$ given $X_{n+1} \in I^i_{j^*}$, where $j^* = 1, \ldots, s_j$, by assigning all conditional M-function values within the interval $I^i = (x_i, x_{i+1})$ to the right of t^i_a, using Equation (34), which leads to the conditional probability mass $P_{X_{n+2}\in I^i|X_{n+1}\in I^i}$.

As a result, the NPI upper probability for the event that $X_{n+1} > t$ and $X_{n+2} > t$, for $t \in [x_i, x_{i+1})$, $i = 0, 1, \ldots, u$, and for all $t > 0$, is obtained by summing the values of $J_1, \ldots, J_4$, i. e. $\overline{P}(X_{n+1} > t, X_{n+2} > t) = J_1 + J_2 + J_3 + J_4$. $\square$

Next, we derive the NPI lower probability for the event that $X_{n+1} > t$ and $X_{n+2} > t$, for $t \in [t^i_a, t^i_{a+1})$, $i = 0, 1, \dots, u$ and $a = 0, 1, \dots, s_i$.

Theorem 5.2. This NPI lower probability is derived by summing all probability masses that must be assigned to the right of t^i_{a+1}. This means all M-function values assigned to intervals $I^k_{k^*}, I^j_{j^*} \in \{I^i_{a+1}, \dots, I^i_{s_i}, I^{i+1}_0, \dots, I^{i+1}_{s_{i+1}}, \dots, I^u_{s_u}\}$. This leads to

$$\underline{P}(X_{n+1} > t, X_{n+2} > t) = \sum_{j=i}^{u} \sum_{j^*=a+1}^{s_j} \sum_{k=i}^{u} \sum_{k^*=a+1}^{s_k} M_{X_{n+2} \in I^k_{k^*} | X_{n+1} \in I^j_{j^*}} M_{X_{n+1} \in I^j_{j^*}} \tag{53}$$

where we must start from $a+1$; that is, we start from the first right-censored observation up to s_i within the interval I^i.

Proof. There are four terms of summations that, when added together, lead to derive the NPI lower probability for the event $X_{n+1} > t$ and $X_{n+2} > t$. We refer to these terms as K_1, K_2, K_3, and K_4, stated in Equations (54), (55), (56), and (57), respectively, which are illustrated in detail below.

First, similar to the summations in the derivation of the NPI upper probability for this event, given in Equation (49), we sum over $I^k_{k^*} \in \{I^{i+1}_0, \dots, I^{i+1}_{s_{i+1}}, \dots, I^u_{s_u}\}$ and $I^j_{j^*} \in \{I^{i+1}_0, \dots, I^{i+1}_{s_{i+1}}, \dots, I^u_{s_u}\}$, which will lead to constant probabilities using Equations (15) and (29), respectively, so these probabilities are not functions of the α's or β's, so no optimisation is required here. We can write these summations terms as

$$K_1 = J_1 = \sum_{j=i+1}^{u} \sum_{k=i+1}^{u} P_{X_{n+2} \in I^k | X_{n+1} \in I^j} P_{X_{n+1} \in I^j} \tag{54}$$

Secondly, we sum over $I^k_{k^*} \in \{I^i_{a+1}, \dots, I^i_{s_i}\}$ and $I^j_{j^*} \in \{I^{i+1}_0, \dots, I^{i+1}_{s_{i+1}}, \dots, I^u_{s_u}\}$, which will lead to a function of the β's only, so we need to minimise this function. This leads to

$$\begin{aligned}
K_2 &= \sum_{j=i+1}^{u} \sum_{j^*=0}^{s_j} \sum_{k^*=a+1}^{s_i} M_{X_{n+2} \in I^i_{k^*} | X_{n+1} \in I^j_{j^*}} M_{X_{n+1} \in I^j_{j^*}} \\
&= \sum_{j=i+1}^{u} \sum_{j^*=0}^{s_j} M_{X_{n+1} \in I^j_{j^*}} \sum_{k^*=a+1}^{s_i} M_{X_{n+2} \in I^i_{k^*} | X_{n+1} \in I^j_{j^*}} \\
&= \sum_{j=i+1}^{u} \sum_{j^*=0}^{s_j} M_{X_{n+1} \in I^j_{j^*}} Q^{\min}_{X_{n+2} \in I^i_a | X_{n+1} \in I^j_{j^*}} \\
&= \sum_{j=i+1}^{u} Q^{\min}_{X_{n+2} \in I^i_a | X_{n+1} \in I^j} P_{X_{n+1} \in I^j}
\end{aligned} \tag{55}$$

where in the third equality, the function $Q^{\min}_{X_{n+2} \in I^i_a | X_{n+1} \in I^j_{j^*}}$ is considered to minimise the conditional probability mass for $X_{n+2} \in I^i_a = (t^i_a, x_{i+1})$ given $X_{n+1} \in I^j_{j^*}$, where $j = i +$

$1,\ldots,u$, by assigning all conditional M-function values within the interval $I^i = (x_i, x_{i+1})$ to the left of t_a^i. This leads to the conditional probability mass $Q^{\min}_{X_{n+2}\in I_a^i|X_{n+1}\in I^j}$; that is we sum all the conditional probability mass for the event $X_{n+2} \in (c_{i^*}^i, x_{i+1})$ given $X_{n+1} \in I^j$, where $i = 0,1,2,\ldots,u$, $i^* = 1,2,\ldots,s_i$ and $j = i+1,\ldots,u$. Then, we are able to move from the third equality to the fourth equality via the product of the conditional probability $Q^{\min}_{X_{n+2}\in I_a^i|X_{n+1}\in I^j}$, and the probability masses for the event that $X_{n+1} \in I^j$. The function $Q^{\min}_{X_{n+2}\in I_a^i|X_{n+1}\in I_{j^*}^j}$, which is a function of the β's only, is minimised by using Equation (33).

Thirdly, we sum over $I_{k^*}^k \in \{I_0^{i+1},\ldots,I_{s_{i+1}}^{i+1},\ldots,I_{s_u}^u\}$ and $I_{j^*}^j \in \{I_{a+1}^i,\ldots,I_{s_i}^i\}$, which will lead to a function of the α's only, so we need to minimise this function. This leads to

$$
\begin{aligned}
K_3 &= \sum_{k=i+1}^{u} \sum_{j^*=a+1}^{s_i} \sum_{k^*=0}^{s_k} M_{X_{n+2}\in I_{k^*}^k|X_{n+1}\in I_{j^*}^i} M_{X_{n+1}\in I_{j^*}^i} \\
&= \sum_{k=i+1}^{u} \sum_{j^*=a+1}^{s_i} P_{X_{n+2}\in I^k|X_{n+1}\in I_{j^*}^i} M_{X_{n+1}\in I_{j^*}^i} \\
&= \sum_{k=i+1}^{u} P_{X_{n+2}\in I^k|X_{n+1}\in I_a^i} Q^{\min}_{X_{n+1}\in I_a^i} \qquad (56)
\end{aligned}
$$

where in the third equality, the function $Q^{\min}_{X_{n+1}\in I_a^i}$ is considered to minimise the probability mass for $X_{n+1} \in I_a^i = (t_a^i, x_{i+1})$, by assigning all M-function values within the interval $I^j = (x_j, x_{j+1})$ to the left of t_a^i. This leads to the probability mass for the event $X_{n+1} \in (c_{i^*}^i, x_{i+1})$, using Equation (19). This enables us to obtain the product of the conditional probabilities for the event that $X_{n+2} \in I^k|X_{n+1} \in I_a^i$, and the probability mass $Q^{\min}_{X_{n+1}\in I_a^i}$, where $k = i+1,\ldots,u$ and $I_a^i = (t_a^i, x_{i+1})$.

Finally, we sum over $I_{k^*}^k \in \{I_{a+1}^i,\ldots,I_{s_i}^i\}$ and $I_{j^*}^j \in \{I_{a+1}^i,\ldots,I_{s_i}^i\}$, which will lead to functions of the α's and β's, so we need to minimise both functions. This leads to

$$
\begin{aligned}
K_4 &= \sum_{j^*=a+1}^{s_i} \sum_{k^*=a+1}^{s_i} M_{X_{n+2}\in I_{k^*}^i|X_{n+1}\in I_{j^*}^i} M_{X_{n+1}\in I_{j^*}^i} \\
&= Q^{\min}_{X_{n+2}\in I_a^i|X_{n+1}\in I_a^i} Q^{\min}_{X_{n+1}\in I_a^i} \qquad (57)
\end{aligned}
$$

where in the second equality, the function $Q^{\min}_{X_{n+1}\in I_a^i}$ is considered to minimise the probability mass for $X_{n+1} \in I_a^i = (t_a^i, x_{i+1})$, by assigning all M-function values within the interval $I^i = (x_i, x_{i+1})$ to the left of t_a^i, using Equation (19). Also, the function $Q^{\min}_{X_{n+2}\in I_a^i|X_{n+1}\in I_a^i}$ is considered to minimise the conditional probability mass for $X_{n+2} \in I_a^i = (t_a^i, x_{i+1})$ given $X_{n+1} \in I_a^i$, by assigning all conditional M-function values within the interval $I^i = (x_i, x_{i+1})$ to the left of t_a^i, using Equation (33).

As a result, the NPI lower probability for the event that $X_{n+1} > t$ and $X_{n+2} > t$, for $t \in [t_a^i, t_{a+1}^i)$, $i = 0,1,\ldots,u$ and $a = 0,1,\ldots,s_i$, and for all $t > 0$, is obtained by summing the values of $K_1,\ldots,K_4$, i. e. $\underline{P}(X_{n+1} > t, X_{n+2} > t) = K_1 + K_2 + K_3 + K_4$. □

Using the results in Sections 3 and 4, we also get the same results of the derivation of the NPI lower and upper probabilities for the joint event $X_{n+1} > t$ and $X_{n+2} > t$, presented

	$X_6 \in I^0 = (0, x_1)$	$X_6 \in I_1^1 = (x_1, c_1^1)$	$X_6 \in I_2^1 = (c_1^1, x_2)$	$X_6 \in I^2 = (x_2, x_3)$	$X_6 \in I^3 = (x_3, \infty)$	Total
$X_5 \in I^0 = (0, x_1)$	$2 \cdot \frac{1}{30}$	$\frac{1}{30}\beta_1^2$	$\frac{1}{30}(\beta_2^2 + \frac{1}{3})$	$\frac{1}{30} \cdot \frac{4}{3}$	$\frac{1}{30} \cdot \frac{4}{3}$	$\frac{1}{5}$
$X_5 \in I_1^1 = (x_1, c_1^1)$	$\frac{1}{30}\alpha_1^1$	$\frac{1}{30}\alpha_1^1(1 + \beta_1^2)$	$\frac{1}{30}\alpha_1^1(\beta_2^2 + \frac{1}{3})$	$\frac{1}{30} \cdot \frac{4}{3}\alpha_1^1$	$\frac{1}{30} \cdot \frac{4}{3}\alpha_1^1$	$\frac{1}{5}\alpha_1^1$
$X_5 \in I_2^1 = (c_1^1, x_2)$	$\frac{1}{30}(\alpha_2^1 + \frac{1}{3})$	$\frac{1}{30}\beta_1^1(\alpha_2^1 + \frac{1}{3})$	$\boxed{\frac{1}{30}(\alpha_2^1 + \frac{1}{3})(\beta_2^1 + \frac{3}{2})}$	$\boxed{\frac{1}{30}(\alpha_2^1 + \frac{1}{3})(\frac{5}{4})}$	$\boxed{\frac{1}{30}(\alpha_2^1 + \frac{1}{3})(\frac{5}{4})}$	$\frac{1}{5}\alpha_2^1 + \frac{1}{15}$
$X_5 \in I^2 = (x_2, x_3)$	$\frac{1}{30} \cdot \frac{4}{3}$	$\frac{1}{30} \cdot \frac{4}{3}\beta_1^1$	$\boxed{\frac{1}{30} \cdot \frac{4}{3}(\beta_2^1 + \frac{1}{4})}$	$\boxed{\frac{1}{30} \cdot \frac{10}{3}}$	$\boxed{\frac{1}{30} \cdot \frac{5}{3}}$	$\frac{4}{15}$
$X_5 \in I^3 = (x_3, \infty)$	$\frac{1}{30} \cdot \frac{4}{3}$	$\frac{1}{30} \cdot \frac{4}{3}\beta_1^1$	$\boxed{\frac{1}{30} \cdot \frac{4}{3}(\beta_2^1 + \frac{1}{4})}$	$\boxed{\frac{1}{30} \cdot \frac{5}{3}}$	$\boxed{\frac{1}{30} \cdot \frac{10}{3}}$	$\frac{4}{15}$

Table 3: Joint probability of X_5 and X_6, according to Example 5.1

above, if we straightforwardly multiply the NPI lower and upper probabilities for the event $X_{n+1} > t$, given by Equations (21) and (22), respectively, in Section 3, with the NPI lower and upper conditional probabilities for the event that $X_{n+2} > t$ given $X_{n+1} > t$, given by Equations (35) and (36), respectively, in Section 4. So, for $t \in [t_a^i, t_{a+1}^i)$ with $i = 0, 1, \ldots, u$ and $a = 0, 1, \ldots, s_i$, the NPI lower probability for the joint event $X_{n+1} > t$ and $X_{n+2} > t$, is

$$\underline{P}(X_{n+1} > t, X_{n+2} > t) = \underline{P}(X_{n+2} > t | X_{n+1} > t)\underline{P}(X_{n+1} > t) \tag{58}$$

and for $t \in [x_i, x_{i+1})$ with $i = 0, 1, \ldots, u$, the corresponding NPI upper probability for the joint event $X_{n+1} > t$ and $X_{n+2} > t$, is

$$\overline{P}(X_{n+1} > t, X_{n+2} > t) = \overline{P}(X_{n+2} > t | X_{n+1} > t)\overline{P}(X_{n+1} > t) \tag{59}$$

Example 5.1 illustrates the NPI lower and upper probabilities for the events $X_5 > t$ and $X_6 > t$, in particular it shows the steps leading to these lower and upper probabilities in Theorems 5.1 and 5.2. It also demonstrates the results in Equations (58) and (59).

Example 5.1. Consider again the data set used in Examples 3.1 and 4.1, for which we have $n = 4$ observations, including one right-censored observation. Based on the probability masses for X_5, presented in Figure 2, and the conditional probability masses for $X_6 | X_5$, in Figure 3, the joint probability masses for X_5 and X_6 are given in Table 3. Note that from Examples 3.1 and 4.1, $\alpha_1^{c_1^1} = 1$, $\beta_1^{c_1^2} = 1$, and $\beta_1^{c_1^1} = 1$.

From Table 3, the upper probability for the event that $X_5 > t$ and $X_6 > t$ when $t \in (c_1^1, x_2)$ can be calculated by summing all probabilities represented by blue cells as well as maximising all probability masses represented by green, purple, and red cells. We refer to these summations terms as J_1, J_2, J_3, and J_4, as given by Equations (49), (50), (51) and (52), respectively. These summations are illustrated in detail below.

Considering $t \in (c_1^1, x_2)$, we first sum over I^2 and I^3, respectively, where X_6 is in intervals I^2, I^3 given X_5 is in these intervals I^2, I^3, respectively. This will lead to constant probabilities which are represented by the blue cells in Table 3, which is not a function of the α's or β's, so no optimisation is required here. These summations are derived by using Equation (49), as

$$\begin{aligned} J_1 &= P_{X_6 \in I^2 | X_5 \in I^2} P_{X_5 \in I^2} + P_{X_6 \in I^2 | X_5 \in I^3} P_{X_5 \in I^3} + P_{X_6 \in I^3 | X_5 \in I^2} P_{X_5 \in I^2} \\ &\quad + P_{X_6 \in I^3 | X_5 \in I^3} P_{X_5 \in I^3} \\ &= \frac{1}{9} + \frac{1}{18} + \frac{1}{18} + \frac{1}{9} = \frac{1}{3}. \end{aligned}$$

By using Equation (50), we sum over the case where X_6 is in interval $I_2^1 = (c_1^1, x_2)$, given X_5 is in intervals I^2 and I^3, respectively, represented by the red cells in Table 3. This will lead to a function of the β's only, so we need to maximise this function. This leads to

$$\begin{aligned}
J_2 &= Q^{\max}_{X_6 \in I_2^1 | X_5 \in I^2} P_{X_5 \in I^2} + Q^{\max}_{X_6 \in I_2^1 | X_5 \in I^3} P_{X_5 \in I^3} \\
&= \frac{2}{45}(\beta_2^1 + \frac{1}{4}) + \frac{2}{45}(\beta_2^1 + \frac{1}{4}) = \frac{2}{45}(2\beta_2^1 + \frac{1}{2})
\end{aligned}$$

Here, by using Equation (34), the function $Q_{X_6 \in I_2^1 | X_5 \in I^i} = 2\beta_2^1 + \frac{1}{2}$, for $i = 2, 3$, which is a function of the β's only, is maximised by assigning all conditional M-function values within the interval $I^1 = (x_1, x_2)$ to the right of c_1^1. This is achieved when $\beta_2^1 = 1$, so $\beta_1^1 = 0$ and $Q^{\max}_{X_6 \in I_2^1 | X_5 \in I^i} = 2 + \frac{1}{2} = \frac{5}{2}$. Consequently, $J_2 = \frac{5}{2} \times \frac{2}{45} = \frac{1}{9}$.

Using Equation (51), we sum over the case where X_6 is in intervals I^2 and I^3, given X_5 is in interval $I_2^1 = (c_1^1, x_2)$, respectively, represented by the purple cells in Table 3. This will lead to a function of the α's only, so we need to maximise this function. This leads to

$$\begin{aligned}
J_3 &= P_{X_6 \in I^2 | X_5 \in I_2^1} Q^{\max}_{X_5 \in I_2^1} + P_{X_6 \in I^3 | X_5 \in I_2^1} Q^{\max}_{X_5 \in I_2^1} \\
&= \frac{1}{24}(\alpha_2^1 + \frac{1}{3}) + \frac{1}{24}(\alpha_2^1 + \frac{1}{3}) = \frac{1}{24}(2\alpha_2^1 + \frac{2}{3})
\end{aligned}$$

Here, the function $Q_{X_5 \in I_2^1} = 2\alpha_2^1 + \frac{2}{3}$, which is a function of the α's only, is maximised, using Equation (20), when $\alpha_2^1 = 1$, so $\alpha_2^1 = 0$ and $Q^{\max}_{X_5 \in I_2^1} = 2 + \frac{2}{3} = \frac{8}{3}$. Consequently, $J_3 = \frac{1}{24} \times \frac{8}{3} = \frac{1}{9}$.

Finally, by using Equation (52), we sum over the case where X_6 given X_5 are both in the interval $I_2^1 = (c_1^1, x_2)$, represented by the green cells in Table 3. This will lead to functions of the α's and β's, so we need to maximise both functions. This leads to

$$\begin{aligned}
J_4 &= Q^{\max}_{X_6 \in I_2^1 | X_5 \in I_2^1} Q^{\max}_{X_5 \in I_2^1} \\
&= \frac{1}{30}(\beta_2^1 + \frac{3}{2})(\alpha_2^1 + \frac{1}{3})
\end{aligned}$$

By using Equation (20), the function $Q_{X_5 \in I_2^1} = \alpha_2^1 + \frac{1}{3}$ is maximised when $\alpha_2^1 = 1$, so $\alpha_1^1 = 0$ and $Q^{max}_{X_5 \in I_2^1} = 1 + \frac{1}{3} = \frac{4}{3}$. And the function $Q_{X_6 \in I_2^1 | X_5 \in I_2^1} = \beta_2^1 + \frac{3}{2}$ is maximised, using Equation (34), by assigning all conditional M-function values within the interval $I^1 = (x_1, x_2)$ to the right of c_1^1, i.e., when $\beta_2^1 = 1$, so $\beta_1^1 = 0$ and $Q^{max}_{X_6 \in I_2^1 | X_5 \in I_2^1} = 1 + \frac{3}{2} = \frac{5}{2}$. Consequently, $J_4 = \frac{1}{30} \times \frac{5}{2} \times \frac{4}{3} = \frac{1}{9}$.

As a result, the NPI upper probability for the events $X_5 > t$ and $X_6 > t$, for $t \in (c_1^1, x_2)$, is obtained by summing $J_1 + J_2 + J_3 + J_4$, that is $1/3 + 1/9 + 1/9 + 1/9 = 2/3$. Thus, the NPI upper probability for the events $X_5 > t$ and $X_6 > t$, where $t \in (c_1^1, x_2)$, is $\overline{P}(X_5 > t, X_6 > t) = \frac{2}{3}$ (see Table 4). The NPI upper probabilities for the events $X_5 > t$ and $X_6 > t$, for t in other intervals are given in Table 4, these have all been derived similarly using corresponding values of $J_1, \ldots, J_4$.

The NPI lower probability for the event that $X_5 > t$ and $X_6 > t$ when $t \in (x_1, c_1^1)$ can be calculated by summing all probabilities represented by blue cells as well as minimising all probability masses represented by green, purple, and red cells of the Table 3. We refer to

$t \in (.)$	$\underline{P}(X_5 > t, X_6 > t)$	$\overline{P}(X_5 > t, X_6 > t)$
$(0, x_1)$	$\frac{2}{3}$	1
(x_1, c_1^1)	$\frac{2}{5}$	$\frac{2}{3}$
(c_1^1, x_2)	$\frac{1}{3}$	$\frac{2}{3}$
(x_2, x_3)	$\frac{1}{9}$	$\frac{1}{3}$
(x_3, ∞)	0	$\frac{1}{9}$

Table 4: NPI lower and upper probabilities for the event $(X_5 > t, X_6 > t)$, Example 5.1.

these summations terms as K_1, K_2, K_3 and K_4, as given by Equations (54), (55), (56) and (57), respectively. These summations are illustrated in detail below.

First, and similar to the summation of the upper case which was represented by Equation (49), we sum over I^2 and I^3, respectively, where X_6 is in intervals I^2, I^3 given X_5 is in these intervals I^2, I^3, respectively. This will lead to constant probabilities which are represented by the blue cells in Table 3, which is not a function of the α's or β's, so no optimisation is required here. These summations are derived by using Equation (54), that is $K_1 = J_1 = 1/3$.

By using Equation (55), we sum over the case where X_6 is in interval I_2^1, given X_5 is in intervals I^2 and I^3, respectively, represented by the red cells in Table 3. This will lead to a function of the β's only, so we need to minimise this function. This leads to

$$\begin{aligned} K_2 &= Q^{min}_{X_6 \in I_2^1 | X_5 \in I^2} P_{X_5 \in I^2} + Q^{min}_{X_6 \in I_2^1 | X_5 \in I^3} P_{X_5 \in I^3} \\ &= \frac{2}{45}(2\beta_2^1 + \frac{1}{2}) \end{aligned}$$

Here, by using Equation (33), the function $Q_{X_6 \in I_2^1 | X_5 \in I^2} = 2\beta_2^1 + \frac{1}{2}$, for $i = 2, 3$, which is a function of the β's only, is minimised by assigning all conditional M-function values within the interval $I^1 = (x_1, x_2)$ to the left of c_1^1. This is achieved when $\beta_2^1 = 0$, so $\beta_1^1 = 1$ and $Q^{min}_{X_6 \in I_2^1 | X_5 \in I^2} = \frac{1}{2}$. Consequently, $K_2 = \frac{2}{45} \times \frac{1}{2} = \frac{1}{45}$.

Using Equation (56), we sum over the case where X_6 is in intervals I^2 and I^2, given X_5 is in interval I_2^1, respectively, represented by the purple cells in Table 3. This will lead to a function of the α's only, so we need to minimise this function. This leads to

$$\begin{aligned} K_3 &= P_{X_6 \in I^2 | X_5 \in I_2^1} Q^{min}_{X_5 \in I_2^1} + P_{X_6 \in I^3 | X_5 \in I_2^1} Q^{min}_{X_5 \in I_2^1} \\ &= \frac{1}{24}(2\alpha_2^1 + \frac{2}{3}) \end{aligned}$$

Here, the function $Q_{X_5 \in I_2^1} = 2\alpha_2^1 + \frac{2}{3}$, which is a function of the α's only, is minimised, using Equation (19), when $\alpha_2^1 = 0$, so $\alpha_1^1 = 1$ and $Q^{min}_{X_5 \in I_2^1} = \frac{2}{3}$. Consequently, $K_3 = \frac{1}{24} \times \frac{2}{3} = \frac{1}{36}$.

Finally, by using Equation (57), we sum over the case where X_6 given X_5 are both in interval $I_2^1 = (c_1^1, x_2)$, represented by the green cells in Table 3. This will lead to functions of the α's and β's, so we need to minimise both functions. This leads to

$$\begin{aligned} K_4 &= Q^{min}_{X_6 \in I_2^1 | X_5 \in I_2^1} Q^{min}_{X_5 \in I_2^1} \\ &= \frac{1}{30}(\beta_2^1 + \frac{3}{2})(\alpha_2^1 + \frac{1}{3}) \end{aligned}$$

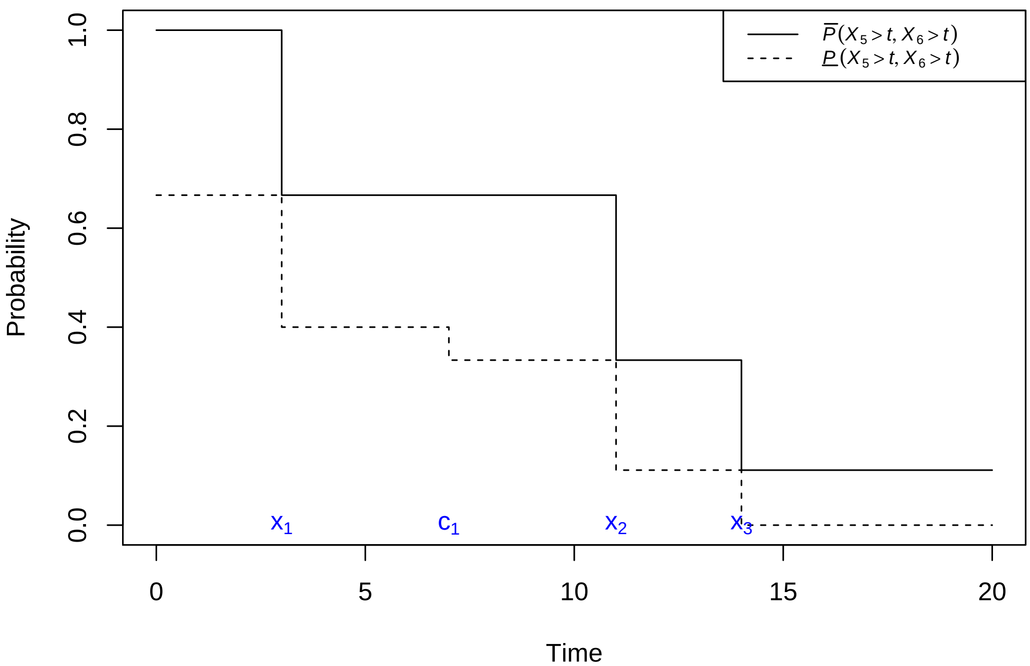

Figure 4: NPI lower and upper probabilities for event $X_5 > t$ and $X_6 > t$, Example 5.1.

$t \in (.)$	$\underline{P}(X_5 > t)$	$\overline{P}(X_5 > t)$	$\underline{P}(X_6 > t \mid X_5 > t)$	$\overline{P}(X_6 > t \mid X_5 > t)$	$\underline{P}(X_5 > t, X_6 > t)$	$\overline{P}(X_5 > t, X_6 > t)$
$(0, x_1)$	$\frac{4}{5}$	1	$\frac{5}{6}$	1	$\frac{2}{3}$	1
(x_1, c_1^1)	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{2}{5}$	$\frac{2}{3}$
(c_1^1, x_2)	$\frac{8}{15}$	$\frac{4}{5}$	$\frac{5}{8}$	$\frac{5}{6}$	$\frac{1}{3}$	$\frac{2}{3}$
(x_2, x_3)	$\frac{4}{15}$	$\frac{8}{15}$	$\frac{5}{12}$	$\frac{5}{8}$	$\frac{1}{9}$	$\frac{1}{3}$
(x_3, ∞)	0	$\frac{4}{15}$	$\frac{5}{24}$	$\frac{5}{12}$	0	$\frac{1}{9}$

Table 5: NPI lower and upper probabilities for the events $(X_5 > t)$, $(X_6 > t|X_5 > t)$ and $(X_5 > t, X_6 > t)$, Example 5.1.

By using Equation (19), the function $Q_{X_5 \in I_2^1} = \alpha_2^1 + \frac{1}{3}$ is minimised when $\alpha_2^1 = 0$, so $\alpha_1^1 = 1$ and $Q^{min}_{X_5 \in I_2^1} = \frac{1}{3}$. And the function $Q_{X_6 \in I_2^1 | X_5 \in I_2^1} = \beta_2^1 + \frac{3}{2}$ is minimised, using Equation (33), by assigning all conditional M-function values within the interval $I^1 = (x_1, x_2)$ to the left of c_1^1, i.e., when $\beta_2^1 = 0$, so $\beta_1^1 = 1$ and $Q^{min}_{X_6 \in I_2^1 | X_5 \in I_2^1} = \frac{3}{2}$. Consequently, $K_4 = \frac{1}{30} \times \frac{3}{2} \times \frac{1}{3} = \frac{1}{60}$.

As a result, the NPI lower probability for the events $X_5 > t$ and $X_6 > t$, for $t \in (x_1, c_1^1)$, is obtained by summing $K_1 + K_2 + K_3 + K_4$, that is $1/3 + /45 + /36 + 1/60 = 2/5$. Thus, the NPI lower probability for the events $X_5 > t$ and $X_6 > t$, where $t \in (x_1, c_1^1)$, is $\underline{P}(X_5 > t, X_6 > t) = \frac{2}{5}$ (see Table 4). The NPI lower probabilities for the events $X_5 > t$ and $X_6 > t$, for t in other intervals are given in Table 4, and shown in Figure 4, these have all been derived similarly using corresponding values of $K_1, \ldots, K_4$.

Finally, if we multiply the results of the NPI lower and upper probabilities for the event $X_5 > t$, presented in Table 1, with the corresponding results of the NPI lower and upper conditional probabilities for the event $X_6 > t|X_5 > t$, presented in Table 2, then we get the same results of the NPI lower and upper probabilities for the joint event that $X_5 > t$

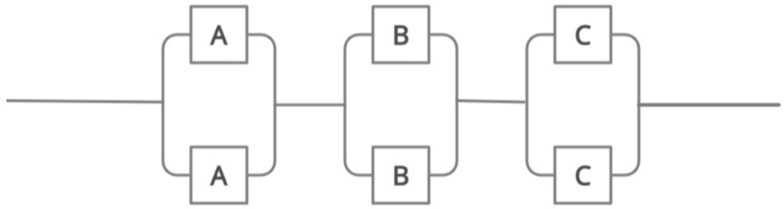

Figure 5: A series system with three types of components A, B and C, with two components of each type.

and $X_6 > t$, shown in Table 4 and Figure 4. The point is clearly illustrated by the results presented in Table 5, see also Equations (58) and (59).

6. Reliability of a series system

This section illustrates how the proposed method can be applied to the reliability of a series system. The application focuses on a series system comprising three pairs of parallel components, as illustrated in Figure 5. Each parallel pair consists of components belonging to types A, B, or C. For each type, 20 components were tested, leading to the observed failure times and right-censoring times presented in Table 6. The failure times of components of different types are assumed to be independent, while failure times of components of the same type are assumed to be exchangeable. Right-censoring is assumed to be non-informative with regard to the component's remaining time to failure.

First we need to introduce some notations. Let m_A, m_B and m_C represent the number of components of Type A, B and C, respectively, so $m_A = m_B = m_C = 2$, with 20 observations for each type, so $n_A = n_B = n_C = 20$. In addition, let $X^A_{i,1}$ and $X^A_{i,2}$, for $i = 1, 2, \dots, n_A$, $X^B_{i,1}$ and $X^B_{i,2}$, for $i = 1, 2, \dots, n_B$, and $X^C_{i,1}$ and $X^C_{i,2}$, for $i = 1, 2, \dots, n_C$, represent the two components of types A, B, C, respectively. Let $T^A_{n_A}$, $T^B_{n_B}$ and $T^C_{n_C}$ represent the minimum of the two components in Types A, B, and C, respectively, e.g. $T^A_{n_A} = \min(X^A_{i,1}, X^A_{i,2})$, etc. Let $\underline{P}_{T^A_{n_A+1},T^A_{n_A+2}}(t)$ and $\overline{P}_{T^A_{n_A+1},T^A_{n_A+2}}(t)$ denote the NPI lower and upper probabilities for the event that the two future failure times of components of Type A are both greater than t, with similar notation for Types B and C.

The data for the components failure and right-censoring times, presented in Table 6, are obtained via simulation. For each component of Type A, 20 failure times are simulated from the Weibull distribution with shape parameter 1.5 and scale parameter 1. Next, the minimum of these two components is obtained, that is $T^A_{20} = \min(X^A_{i,1}, X^A_{i,2})$, for $i = 1, 2, \dots, 20$. For each component of Type B, $X^B_{i,1}$ and $X^B_{i,2}$, 17 failure times, and three right-censoring times are simulated from the Weibull distribution with shape parameter 2 and scale parameter 1 and the exponential distribution with a rate of 0.27, respectively. Now, the minimum of these two components is obtained, that is $T^B_{20} = \min(X^B_{i,1}, X^B_{i,2})$, for $i = 1, 2, \dots, 20$. Also, for each component of Type C, that are $X^C_{i,1}$ and $X^C_{i,2}$, 13 failure times and seven right-censoring times are simulated from the Weibull distribution with shape parameter 3 and scale parameter 1 and the exponential distribution with rate of 0.35, respectively. Next, the minimum of these two components is obtained, that is $T^C_{20} = \min(X^C_{i,1}, X^C_{i,2})$, for $i = 1, 2, \dots, 20$.

In order to compute the reliability of the system for the data set shown in Table 6, the

T^A_{20}		T^B_{20}		T^C_{20}	
0.090	0.461	0.115	0.490	$>$ 0.050	0.593
0.147	0.464	$>$ 0.150	0.496	$>$ 0.161	0.602
0.216	0.472	0.185	0.533	$>$ 0.172	0.604
0.224	0.536	$>$ 0.262	$>$ 0.630	0.257	0.607
0.332	0.552	0.343	0.640	$>$ 0.349	0.693
0.342	0.786	0.401	0.647	0.377	0.728
0.356	0.903	0.421	0.654	$>$ 0.421	0.750
0.377	0.937	0.437	0.729	0.522	0.957
0.388	1.036	0.442	0.852	$>$ 0.539	0.966
0.431	1.400	0.450	1.282	0.563	$>$ 0.976

Table 6: Simulated data with the three types of components A, B and C ($>$ indicates a right-censored observation).

results presented in Section 5 will be first applied separately for each type of component T^A_{20}, T^B_{20} and T^C_{20}. As a result, we obtain the NPI lower and upper probabilities for the event that both future failure times of components from each type exceed t. These probabilities are summarized in Table 7 and depicted in Figure 6.

For Type A, we derive the NPI lower and upper probabilities, that are $[\underline{P}, \overline{P}](T^A_{21} > t_A, T^A_{22} > t_A)$, for $t_A \in (0, \mathit{data(A)}, \infty)$ and for Type B, we derive the NPI lower and upper probabilities, that are $[\underline{P}, \overline{P}](T^B_{21} > t_B, T^B_{22} > t_B)$, for $t_B \in (0, \mathit{data(B)}, \infty)$, and finally for Type C, we derive the NPI lower and upper probabilities, that are $[\underline{P}, \overline{P}](T^C_{21} > t_C, T^C_{22} > t_C)$, for $t_C \in (0, \mathit{data(C)}, \infty)$.

Second, the reliability function of the whole system are derived by multiplying the corresponding intersection NPI lower and upper probabilities for each type presented in Table 7, with the emphasis that the exact values of the t's in this table differ for the different systems. The NPI lower and upper probabilities for the whole reliability system at time t, denoted as $\underline{P}_{T^S_{21}}(t)$ and $\overline{P}_{T^S_{21}}(t)$, respectively, are shown in Figure 7. So the reliability function of the whole system is calculated as $[\underline{P}, \overline{P}](T^S_{21} > t, T^C_{22} > t) = [\underline{P}, \overline{P}](T^A_{21} > t_A, T^A_{22} > t_A) \times [\underline{P}, \overline{P}](T^B_{21} > t_B, T^B_{22} > t_B) \times [\underline{P}, \overline{P}](T^C_{21} > t_C, T^C_{22} > t_C)$, for $t \in (0, \mathit{data}, \infty)$.

It is worth mentioning that the NPI for the joint event $X_{n+1} > t$ and $X_{n+2} > t$, presented in this paper, takes into account the dependence between these two variables when there is limited information in the form of n observations in the data. It is of interest to see the effect of taking this dependence carefully into account. For this reason, we will compare the results followed the proposed method with those resulting from ignoring, mistakenly, the dependency between these two future observations, i.e., one would use the squared NPI lower and upper probabilities for the event $X_{n+1} > t$. Next, we compare the results of the proposed method with those that would occur if we ignored the dependence between the two future observations. And $(\underline{P}_{T^S_{21}}(t))^2$ and $(\overline{P}_{T^S_{21}}(t))^2$ represent the NPI lower and upper probabilities based on the wrong assumption of independence of the two future observations per type of component, as shown in Figure 7.

Figure 7 shows that the proposed method provides lower and upper probabilities $\underline{P}_{T^S_{21}}(t)$ and $\overline{P}_{T^S_{21}}(t)$ of the system failure time, that are never smaller than the incorrect ones via the

$t \in$	$\underline{P}_{T_{21}^A,T_{22}^A}(t)$	$\overline{P}_{T_{21}^A,T_{22}^A}(t)$	$\underline{P}_{T_{21}^B,T_{22}^B}(t)$	$\overline{P}_{T_{21}^B,T_{22}^B}(t)$	$\underline{P}_{T_{21}^C,T_{22}^C}(t)$	$\overline{P}_{T_{21}^C,T_{22}^C}(t)$
$(0, t_1)$	0.909	1	0.909	1	0.909	1
(t_1, t_2)	0.823	0.909	0.823	0.909	0.905	1
(t_2, t_3)	0.740	0.823	0.818	0.909	0.900	1
(t_3, t_4)	0.662	0.740	0.732	0.818	0.895	1
(t_4, t_5)	0.589	0.662	0.727	0.818	0.795	0.895
(t_5, t_6)	0.520	0.589	0.642	0.727	0.789	0.895
(t_6, t_7)	0.455	0.520	0.561	0.642	0.691	0.789
(t_7, t_8)	0.394	0.455	0.487	0.561	0.684	0.789
(t_8, t_9)	0.338	0.394	0.417	0.487	0.586	0.684
(t_9, t_{10})	0.286	0.338	0.353	0.417	0.579	0.684
(t_{10}, t_{11})	0.238	0.286	0.294	0.353	0.482	0.579
(t_{11}, t_{12})	0.195	0.238	0.241	0.294	0.395	0.482
(t_{12}, t_{13})	0.156	0.195	0.193	0.241	0.316	0.395
(t_{13}, t_{14})	0.121	0.156	0.150	0.193	0.246	0.316
(t_{14}, t_{15})	0.091	0.121	0.144	0.193	0.184	0.246
(t_{15}, t_{16})	0.065	0.091	0.103	0.144	0.132	0.184
(t_{16}, t_{17})	0.043	0.065	0.069	0.103	0.088	0.132
(t_{17}, t_{18})	0.026	0.043	0.041	0.069	0.053	0.088
(t_{18}, t_{19})	0.013	0.026	0.021	0.041	0.026	0.053
(t_{19}, t_{20})	0.004	0.013	0.007	0.021	0.009	0.026
(t_{20}, ∞)	0	0.004	0	0.007	0	0.026

Table 7: NPI lower and upper probabilities of Type A, Type B and Type C for the data in Table 6.

squared lower and upper probabilities $(\underline{P}_{T_{21}^S}(t))^2$ and $(\overline{P}_{T_{21}^S}(t))^2$. And they only are equal at the start $(\overline{P}_{T_{21}^S}(t) = (\overline{P}_{T_{21}^S}(t))^2 = 1)$ or at the end $(\underline{P}_{T_{21}^S}(t) = (\underline{P}_{T_{21}^S}(t))^2 = 0)$. While the differences between the lower and upper probabilities may only be small, it should be remarked that for more than two future observations, the differences will be larger. Detailed investigation is left as a topic for future research, as it requires the development of the NPI approach for more than two future observations in case of right-censored data.

7. Concluding remarks

In this paper, we have developed NPI for two future observations in the presence of right-censored data. Specifically, we have considered the event where these two future observations exceed time t. For the first future observation, we have utilized the rc-$A_{(n)}$ assumption [15] without any additional assumptions. Then, we have employed the rc-$A_{(n+1)}$ assumption to establish a partially specified predictive probability distribution for the second future observation conditioned on the first future observation. By employing an analytical approach involving the α's and β's, we have obtained NPI lower and upper probabilities for various events involving the next two future observations. These probabilities have been derived explicitly for the event where both future observations exceed time t, but the method can be extended to encompass general events.

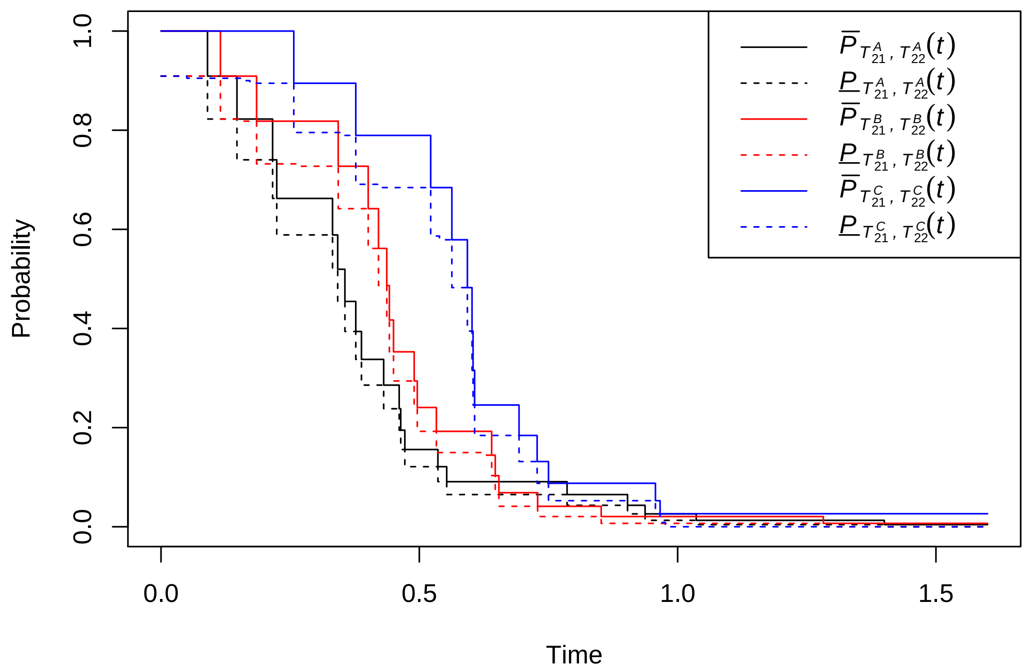

Figure 6: NPI lower and upper probabilities for Types A, B and C of the series system in Table 6.

Through the extension of NPI to two future observations with right-censored data, we have effectively accounted for the dependence between these variables when the available information is limited to n observations in the data. We have compared the results of our proposed method with those obtained by ignoring the dependence between the two future observations. Additionally, we have applied our findings to analyse the system reliability of a small series system comprising three types of components, each consisting of multiple components of the same type. This practical application demonstrates the tangible benefits of our approach.

The joint event of the next two observations both exceeding t, and hence their minimum exceeding t, was considered in this paper, and illustrated through the corresponding reliability function for a series system. A similarly detailed approach could be developed for a parallel system, by considering instead the event that the maximum of the next two observations exceeds t. This, and generalisations to other system reliability scenarios, is left as a topic for future research.

However, we acknowledge that the analytical approach becomes exceedingly complex when dealing with more than two future observations. One possible direction for future research could involve sampling the first future observation using the M-function values, as given by Equations (11) and (12), for X_{n+1}, along with an assumption regarding the distribution of these probabilities within the intervals. Subsequently, this sampled future observation can be added to the dataset, and the process can be repeated for the next observation. The resulting inferences will depend on the assumed distribution per interval, but the computational aspects of this approach are straightforward. This approach aligns with the NPI Bootstrap method [38, 39] and smoothed bootstrap for right-censored data [40].

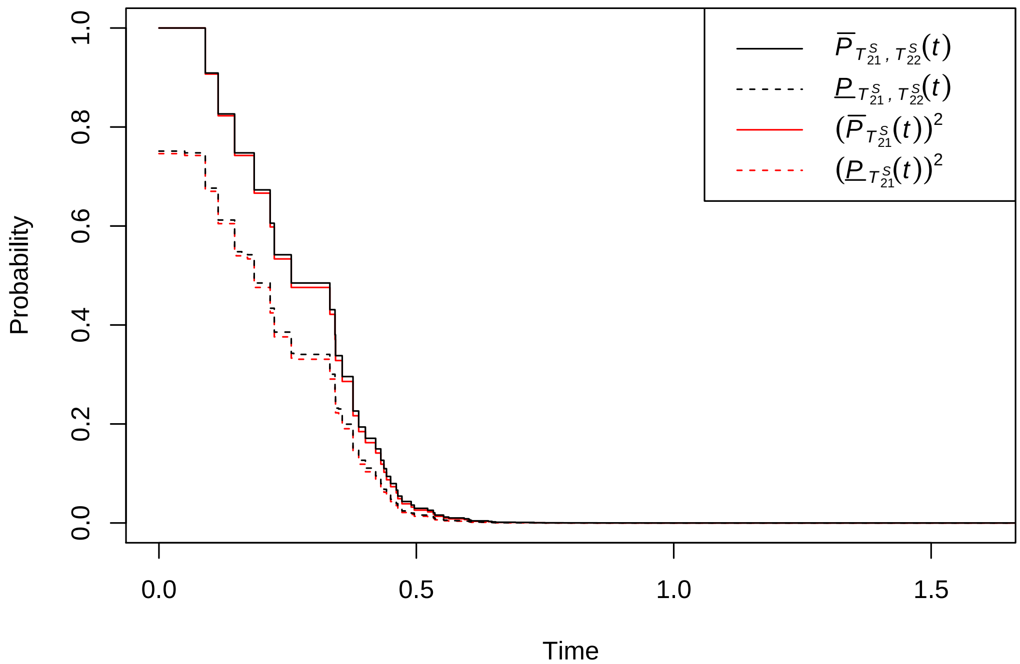

Figure 7: NPI lower and upper probabilities for the whole series system.

In summary, the work presented in this paper shows great potential for extension into various applications, although it is not without its share of mathematical challenges.

Acknowledgements

Ali M. Mahnashi wishes to express his sincere gratitude to Jazan University in Saudi Arabia for providing financial support for the publication of this research paper. The authors would like to express their gratitude to the reviewer for the positive comments on the paper and for providing some useful suggestions for presentation.

References

[1] S. W. Lagakos, General right censoring and its impact on the analysis of survival data, Biometrics (1979) 139–156.

[2] B. M. Hill, Posterior distribution of percentiles: Bayes' theorem for sampling from a population, Journal of the American Statistical Association (1968) 677–691.

[3] B. M. Hill, De finetti's theorem, induction, and bayesian nonparametric predictive inference (with discussion), in: Bayesian Statistics, Oxford University Press, 1988, pp. 211–241.

[4] A. F. Peressini, Imprecise probability and chance, Erkenntnis 81 (2016) 561–586.

[5] P. Walley, Statistical reasoning with imprecise probabilities, Chapman and Hall: London (1991).

[6] H. E. Kyburg, Interval-valued probabilities, Imprecise Probabilities Project, Citeseer (1998).

[7] K. Weichselberger, The theory of interval-probability as a unifying concept for uncertainty, International Journal of Approximate Reasoning 24 (2000) 149–170.

[8] T. Augustin, F. P. A. Coolen, G. De Cooman, M. C. Troffaes, Introduction to Imprecise Probabilities, Wiley, Chichester, 2014.

[9] T. Augustin, F. P. A. Coolen, Nonparametric predictive inference and interval probability, Journal of Statistical Planning and Inference 124 (2004) 251–272.

[10] F. P. A. Coolen, Low structure imprecise predictive inference for bayes' problem, Statistics & Probability Letters 36 (1998) 349–357.

[11] F. P. A. Coolen, P. Coolen-Schrijner, Nonparametric predictive subset selection for proportions, Statistics & Probability Letters 76 (2006) 1675–1684.

[12] F. P. A. Coolen, T. A. Maturi, Nonparametric predictive inference for order statistics of future observations, in: Combining Soft Computing and Statistical Methods in Data Analysis, Proceedings 5th International Conference on Soft Methods in Probability and Statistics, Oviedo (Spain), Sept 2010. Springer, Berlin, 2010, pp. 97–104.

[13] F. P. A. Coolen, P. Van der Laan, Imprecise predictive selection based on low structure assumptions, Journal of Statistical Planning and Inference 98 (2001) 259–277.

[14] T. A. Maturi, P. Coolen-Schrijner, F. P. A. Coolen, Nonparametric predictive pairwise comparison for real-valued data with terminated tails, International Journal of Approximate Reasoning 51 (2009) 141–150.

[15] F. P. A. Coolen, K. J. Yan, Nonparametric predictive inference with right-censored data, Journal of Statistical Planning & Inference 126 (2004) 25–54.

[16] F. P. A. Coolen, K. J. Yan, Nonparametric predictive comparison of two groups of lifetime data., in: Proceedings of the Third International Symposium on Imprecise Probabilities and Their Applications, (eds J.M. Bernard, T. Seidenfeld and M. Zaffalon) Carlton Scientific, 2003, pp. 148–161.

[17] T. A. Coolen-Maturi, F. P. A. Coolen, N. Muhammad, Predictive inference for bivariate data: Combining nonparametric predictive inference for marginals with an estimated copula, Journal of Statistical Theory and Practice 10 (2016) 515–538.

[18] R. Baker, Multinomial nonparametric predictive inference: selection, classification and subcategory data, PhD thesis, University of Durham (2010).
URL `http://etheses.dur.ac.uk/257/`

[19] F. P. A. Coolen, T. Augustin, Learning from multinomial data: a nonparametric predictive alternative to the imprecise dirichlet model., in: ISIPTA, Proceedings of the Fourth International Symposium on Imprecise Probabilities and Their Applications, F.G. Cozman, R. Nau and T. Seidenfeld (Eds), 2005, pp. 125–134.

[20] F. P. A. Coolen, On nonparametric predictive inference and objective bayesianism, Journal of Logic, Language & Information 15 (2006) 21–47.

[21] A. M. Aboalkhair, F. P. A. Coolen, I. M. MacPhee, Nonparametric predictive inference for reliability of a k-out-of-m: G system with multiple component types, Reliability Engineering & System Safety 131 (2014) 298–304.

[22] F. P. A. Coolen, P. Coolen-Schrijner, A nonparametric predictive method for queues, European Journal of Operational Research 145 (2003) 425–442.

[23] K. Janurová, R. Briš, A nonparametric approach to medical survival data: Uncertainty in the context of risk in mortality analysis, Reliability Engineering & System Safety 125 (2014) 145–152.

[24] B. M. Hill, Parametric models for a(n): splitting processes and mixtures, Journal of the Royal Statistical Society. Series B (1993) 423–433.

[25] B. M. Hill, Bayesian nonparametric prediction and statistical inference, in: Bayesian Analysis in Statistics and Econometrics, Springer, New York, NY, 1992, pp. 43–94.

[26] B. De Finetti, Theory of Probability: A Critical Introductory Treatment, Wiley, London, 1974.

[27] K. Weichselberger, Elementare Grundbegriffe einer Allgemeineren Wahrscheinlichkeitsrechnung I. Intervall-wahrscheinlichkeit als Umfassendes Konzept., Physika, Heidelberg, 2001.

[28] G. Boole, An Investigation of the Laws of Thought: On which are Founded the Mathematical Theories of Logic and Probabilities, Vol. 2, Walton and Maberly, Dover., 1854.

[29] K. Weichselberger, Axiomatic foundations of the theory of intervalprobability. in: V. mammitzsch and h. schneeweiss (eds.), Proceedings of the Second Gauβ Symposion, Section B (1995) 47–64.

[30] A. M. Y. Mahnashi, Contributions to nonparametric predictive inference with right-censored data, PhD thesis, University of Durham (2022).
URL http://etheses.dur.ac.uk/14548/

[31] L. M. Berliner, B. M. Hill, Bayesian nonparametric survival analysis, Journal of the American Statistical Association 83 (1988) 772–779.

[32] T. Coolen-Maturi, F. P. A. Coolen, Nonparametric predictive inference for combined competing risks data, Reliability Engineering & System Safety 126 (2014) 87–97.

[33] G. Shafer, A Mathematical Theory of Evidence, Princeton University Press, New Jersey, 1976.

[34] T. A. Maturi, Nonparametric predictive inference for multiple comparisons, PhD thesis, University of Durham (2010).
URL http://etheses.dur.ac.uk/230/

[35] K. J. Yan, Nonparametric predictive inference with right-censored data, PhD Thesis, Durham University (2002).
URL http://etheses.dur.ac.uk/4026/

[36] G. R. J. Arts, F. P. A. Coolen, Two nonparametric predictive control charts, Journal of Statistical Theory and Practice 2 (2008) 499–512.

[37] G. R. J. Arts, F. P. A. Coolen, P. van der Laan, Nonparametric predictive inference in statistical process control, Quality Technology & Quantitative Management 1 (2004) 201–216.

[38] S. BinHimd, Nonparametric predictive methods for bootstrap and test reproducibility, PhD thesis, University of Durham (2014).
URL http://etheses.dur.ac.uk/9493/

[39] F. P. A. Coolen, S. BinHimd, Nonparametric predictive inference bootstrap with application to reproducibility of the two-sample kolmogorov–smirnov test, Journal of Statistical Theory and Practice 14 (2020) 1–13.

[40] A. S. M. Al-Luhayb, Smoothed bootstrap methods for right-censored data and bivariate data, PhD thesis, University of Durham (2021).
URL http://etheses.dur.ac.uk/14096/