Nonparametric Predictive Inference for American Option Pricing based on the Binomial Tree Model

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ABSTRACT

In this paper, we present the American option pricing procedure based on the binomial tree from an imprecise statistical aspect. Nonparametric Predictive Inference (NPI) is implemented to infer imprecise probabilities of underlying asset movements, reflecting uncertainty while learning from data, which is superior to the constant risk-neutral probability. In a recent paper, we applied the NPI method to the European option pricing procedure that gives good results when the investor has non-perfect information. We now investigate the NPI method for American option pricing, of which imprecise probabilities are considered and updated for every one-time-step path. Different from the classic models, this method is shown that it may be optimal to early exercise an American non-dividend call option because our method considers all information that occurs in the future steps. We also study the performance of the NPI pricing method for American options via simulations in two different scenarios compared to the Cox, Ross and Rubinstein binomial tree model(CRR), first where the CRR assumptions are right, and second where the CRR model uses wrong assumptions. Through the performance study, we conclude that the investor using the NPI method tends to achieve good results in the second scenario.

1. INTRODUCTION

In this paper, we present a novel binomial tree pricing method for American options based on the Nonparametric Predictive Inference (NPI), which is a statistical method under few assumptions, with inferences based on the data and uncertainty quantified by lower and upper probabilities. American options give the right of early exercise and are an important type of option in the market. Due to their path dependence feature, it is difficult to find a closed formula for their pricing. The Cox-Ross-Rubinstein Binomial Tree Model (CRR model) by Cox et al. (1979) can be used for American option pricing, and other scholars extend this model to fit more complicated situations. Boyle (1988) set up a binomial tree model for two underlying assets, and Amin (1993) improved the original CRR model by adding jump diffusion, under the assumption of market completeness and risk-neutral world. A binomial tree model with randomized stock price movement is proposed by Hu and Cao (2014). Zdenek (2010) implemented fuzzy set theory in the American real option pricing procedure to quantify uncertainty. However, these models are still under the original assumptions, like the risk-neutral world and constant probability of stock price upward movements, and overlook the information from historical data. A drawback of the CRR model that the stock price movements are assumed to follow the log-normal distribution is attracted a lot attention from many scholars. Nugroho and Morimoto (2016) used the generalized Student's t-error distributions to accommodate the stock return, and Mota and Esquível (2016) proved that the geometric Brownian motion with regimes has a better fit of the stock price than the geometric Brownian motion. However, as it proved by Telmoudi et al. (2016), the parametric procedures for conditional variance modeling lead to model risk, in this sense, the non-parametric model can be considered as an alternative approach.

As the NPI method does the prediction updating with the data with few assumptions, it has been used to solve many issues in finance. NPI has been applied to finance prediction, providing a relatively straightforward way to study future stock return when little further information is available providing an interval probability of the stock return greater than the target return and also a way of the pairwise comparison between stock returns by Baker et al. (2017). The NPI method also be implemented in credit rate for banking based on ROC analysis, which is under few assumptions and uses the imprecise probabilities to qualify the uncertainty by Coolen-Maturi and Coolen (2018). Recently, we presented a novel European option pricing method based on Nonparametric Predictive Inference by He et al. (2019). Instead of using a precise probability for each step in the binomial tree, we applied NPI into this measurement and proved that it is advantageous when the investor has less certain information about the underlying asset. In this paper, we investigate this imprecise statistical method for American options pricing in order to meet the early exercise feature. We apply the NPI method to the discrete binomial tree model to price American options. Comparing to the NPI application for the European option, we add the discount procedure to the option pricing and discuss the influence of the discount factor. As acknowledged, according to rational trading theory states that the American call option without dividends should not be early exercise (Merton, 1973). The validation of this conclusion referring to the NPI method is checked, and the optimal exercise time based on the NPI method is emphatically studied in this paper.

The CRR binomial tree model is structured as follows. Consider an underlying asset with no dividends having initial stock price S_0, which will either go up by the factor $u > 1$ or go down by the factor $0 < d < 1$. There are T life period American options based on this asset with strike price K_c for the call option and K_p for the put option. A constant probability q, $0 < q < 1$, is assumed for each movement to be upward. The pricing formula for a European call option is,

$$V_c^{CRR}[S_T - K_c]^+ = B(0,T) \sum_{k=\lceil k_c^* \rceil}^{T} [u^k d^{T-k} S_0 - K_c] \binom{T}{k} q^k (1-q)^{(T-k)} \tag{1}$$

where $V_c^{CRR}[S_T - K_c]^+$ is the call option expected price calculated from the payoff $[S_T - K_c]^+$ based on the CRR model, k_c^* is such that $u^{k_c^*} d^{T-k_c^*} S_0 - K_c = 0$, and $\lceil k_c^* \rceil$ denotes the smallest integer greater than or equal to k_c^*. $B(0,T)$ is the discount factor during the option life period,

S_T is the expected stock price at the maturity. For a European put option,

$$V_p^{CRR}[K_p - S_T]^+ = B(0,T) \sum_{k=0}^{\lfloor k_p^* \rfloor} [K_p - u^k d^{T-k} S_0] \binom{T}{k} q^k (1-q)^{(T-k)} \tag{2}$$

where $V_p^{CRR}[K_p - S_T]^+$ is the expected payoff of the put option based on the CRR model, k_p^* is such that $K_p - u^{k_p^*} d^{m-k_p^*} S_0 = 0$, and $\lfloor k_p^* \rfloor$ denotes the largest integer less than or equal to k_p^*.

According to the feature of the American option that it can be exercised early, at anytime before its maturity, there is no closed form option pricing formula for American options. Then the values of American options (without any dividend) $V(S,t)$ at time t, with stock price $S_t = S$, are different from European options. In terms of the American option, there is the stopping time τ, when the option is exercised, which for each possible path, the American option is exercised at the optimized stopping time τ giving us the optimization of this American option payoff. Here is the definition of the American option. For an American call option,

$$V_c(S,t) = \max_{\tau} E\left[B(t,\tau)(S_\tau - K_c)^+ \middle| S_t = S\right] \tag{3}$$

$B(t,\tau)$ is the discount factor from t to τ. This formula defines the value of this call option at the time t, as being equal to the discounted instant payoff of this call option at the stopping time τ. For an American put option,

$$V_p(S,t) = \max_{\tau} E\left[B(t,\tau)(K_p - S_\tau)^+ \middle| S_t = S\right] \tag{4}$$

Therefore, the value of this put option at time t is equal to the discounted maximum payoff at τ. There is no closed form formula for the American option pricing. A backward optimization method can be used for the American option pricing, and we apply the NPI to this backward method for the American option pricing.

This paper is organized as follows. Section 2 provides a brief introduction to the NPI method. We propose NPI for American option pricing in Section 3, and in Section 4 we assess the classic rational trading theory, never exercise an American call option with no dividend early, which does not valid from NPI perspective. In Section 5, a performance study via simulations are investigated in two scenarios. Some conclusions and extensions are discussed in Section 6.

2. NONPARAMETRIC PREDICTIVE INFERENCE

In the classic probability theory, for any event of interest A, a precise value $p(A) \in [0,1]$ is deduced to described its probability with this probability p satisfying Kolmogorov's axioms. With the lack of information, the precise probability is not a good way. Instead, the imprecise probability offers an alternative way to investigate event A, qualifying its uncertainty by an interval probabilities with the upper probability $\overline{P}(A)$ and the lower probability $\underline{P}(A)$. And the lower and upper probabilities hold a conjugacy property which links these two probabilities, $P(A) = 1 - P(A^c)$, where A^c is the complementing event of A presented by Augustin and Coolen (2004). Nonparametric Predictive Inference (NPI) is an inferential framework based on the assumption $A_{(n)}$ presented by Hill (1968). It has been developed for Bernoulli data by Coolen (1998), each with a 'success' or 'failure' result. Let Y_1^n represent the number of successful trials in n observed trial. Let Y_n^{n+1} represent the number of successful observations in one future trials. Here the $n+1$ observations are exchangeable. The NPI upper probability for the event $Y_n^{n+1} = 1$, given data $Y_1^n = s$, for $s \in \{0, ..., n\}$, is

$$\overline{P}(Y_n^{n+1} = 1 | Y_1^n = s) = \frac{s+1}{n+1} \tag{5}$$

The corresponding NPI lower probability for this event is,

$$\underline{P}(Y_n^{n+1} = 1 | Y_1^n = s) = \frac{s}{n+1} \tag{6}$$

We can deduce the corresponding NPI lower probabilities by the conjugacy property $\overline{P}(A) = 1 - \underline{P}(A^c)$, where A^c is the complementary event to A,

$$\begin{aligned} \underline{P}(Y_n^{n+1} = 0 | Y_1^n = s) &= 1 - \overline{P}(Y_n^{n+1} = 1 | Y_1^n = s) \\ &= 1 - \frac{s+1}{n+1} = \frac{n-s}{n+1} \end{aligned} \tag{7}$$

$$\begin{aligned} \overline{P}(Y_n^{n+1} = 0 | Y_1^n = s) &= 1 - \underline{P}(Y_{n+1}^{n+m} = 1 | Y_1^n = s) \\ &= 1 - \frac{s}{n+1} = \frac{n+1-s}{n+1} \end{aligned} \tag{8}$$

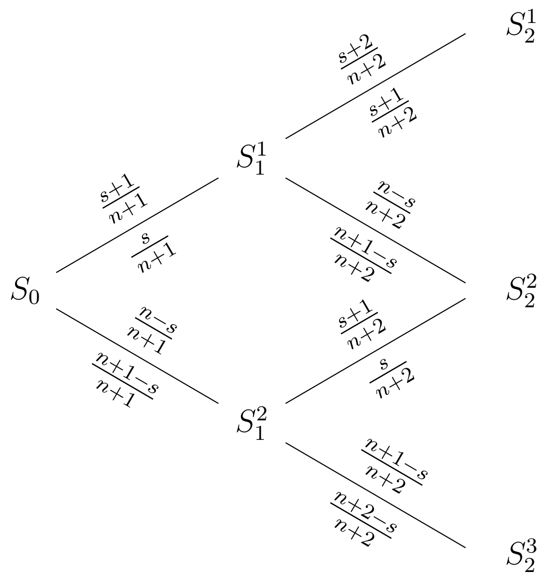


Figure 1: The binomial tree based on the NPI method

In terms of the binomial tree, we could generate upper and lower probabilities for each time step based on the information updating along with the prediction. After predicting upper and lower probabilities at time t, we add the data in the period between $n+1$ to $n+t-1$ to the old historical data getting our new historical data base to predict the next upper and lower probabilities at time $t+1$. Eventually, we will get a binomial tree with imprecise probabilities for each path as presented in Figure 1.

Similarly, NPI can infer the lower and upper expectations of a function $g(Y(m))$ given observed data, here to simplify the formulas we use $Y(m)$ to represent the number of successes in m future trials instead of Y_{n+1}^{n+m}. The NPI method for Bernoulli data by Coolen (1998) provides a set $\mathcal{P}$ of classical, precise, probability distributions for which the presented lower and upper probabilities are optimal bounds. In imprecise probability theory by Augustin et al. (2014), this set $\mathcal{P}$ is called a structure. The lower and upper expected values for a real-valued function g of $Y(m)$ can be derived by

$$\underline{E}(g(Y(m))) = \inf_{p \in \mathcal{P}} E^p(g(Y(m))) \tag{9}$$

$$\overline{E}(g(Y(m))) = \sup_{p \in \mathcal{P}} E^p(g(Y(m))) \tag{10}$$

where E^p is the expected value corresponding to a specific precise probability distribution $p \in \mathcal{P}$. Then for these purposes, we need to use the probability functions that can give us the boundaries of the expected values rather than the probability bounds. Since stock price movements for each time step in the simple Binomial tree model can be represented as Bernoulli data, the NPI for Bernoulli data is suitable to infer imprecise probabilities and expected payoffs for call and put options. As we utilize the NPI method to calculate the expectation of an American option value, we get two prices, namely the NPI lower and upper expected values. We call these the maximum buying price and the minimum selling price, respectively. Because for the investor using our method, when the offered price is higher than the minimum selling price, he is willing to sell the product, while the offered price is lower than the maximum buying price, he is willing to buy the investment product.

3. NPI FOR AMERICAN OPTION PRICING BASED ON THE BINOMIAL TREE MODEL

By the definition of the American option, Equations (3) and (4), it is seen that the discount rate plays an important role in the option pricing procedure. In the CRR binomial tree this discount rate is the risk-free rate r_f, because the CRR binomial tree is settled based on the risk neutral valuation. In a risk neutral world, all products are riskless, and all individuals are indifferent to risk, where their expected return for all products is the risk-free rate. Whereas, in the real world, different investors have different risk levels they can tolerate, so every investment needs to be adjusted according to investors' risk aversion, which is time-consuming and hard to assess precisely. Thus, the risk neutral valuation is commonly used because of its simplicity and efficiency. The discount rate in the risky world is

$$r = r_f + r_{pr} \tag{11}$$

where r_{pr} is the yield of the risk premium referring to the financial product risk. When it refers to underlying assets' discount rate, it equals the risk-free rate plus the risk premium of this asset in this market. However, it is hard to assess the time discount rate for options, because options are riskier than the corresponding stock, and it is not easy to judge the fair risk premium according to the information available from the market by Hull (2009).

Although it is complex to get the discount rate in the real world, there is no reason for us to overlook it. As the discount rate is typically defined as 'the equilibrium expected rate of return on securities equivalent in risk to the project being valued' by Myers (1984), we could use the expected rate of return as the discount rate. In this theoretical study the discount rate is assumed to be equal to the non-negative expected return of the underlying asset.

Due to the early exercise possibility, there is no closed form formula for American options pricing based on the binomial tree model. Thus, the upper and lower probability formulae of the Bernoulli data event can not be used for the American option as it is for the European option by He et al. (2019). Instead, the interval probability for each time step is assigned to address the problem. In Section 1, we provided the definition of American option pricing from the best stopping time aspect. Here we give a different but equivalent definition representing the idea of the backward pricing strategy. Let $h_t(x)$ denote the instant value of the American option at time t, $0 \leq t \leq T$, given $S_t = x$. Then $h_t(x) = x - K_c$ for a call option and $h_t(x) = K_p - x$ for a put option. $V_t(x)$ is the option value at time t given $S_t = x$. The American option value at time t is

$$V_t(x) = \max\{h_t(x), B(t, t+1)V_{t+1}(S_{t+1}|S_t = x)\} \tag{12}$$

$$V_T(x) = \max\{h_T(x), 0\} \tag{13}$$

Here $B(t, t+1)$ is the discount factor between times t and $t+1$. By recursion we compute the value of the American option at the initial time, $V_0(x)$, which is the predicted price of this American option.

3.1 American call option

Figure 2 displays the backward pricing procedure. Suppose there are n historical stock prices available and among them s increased. The call option value in node i at time t is V_t^i. From the tree we could tell then there are $T+1$ levels from level 0 to level T and in each level the number of nodes is the level number plus one, thus $t \in \{0, \ldots, T\}$ and $i \in \{1, \ldots, t+1\}$. We start to evaluate the call option at the maturity $V_T^i = \max\{0, S_T^i - K_c\}$

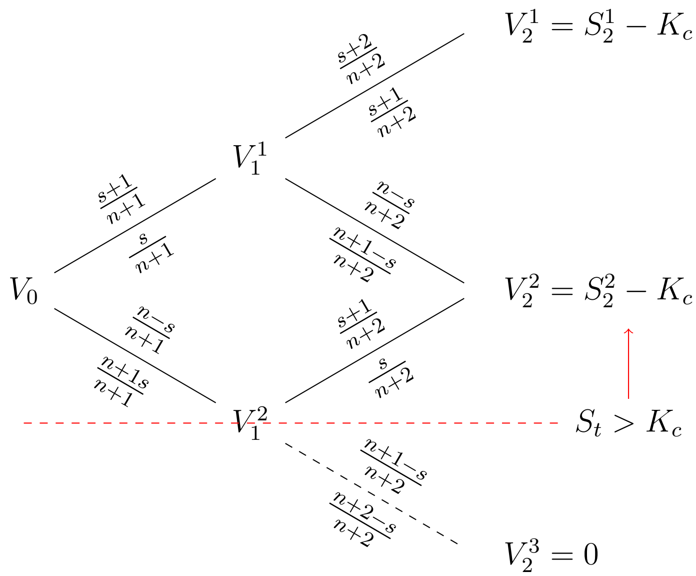


Figure 2: The binomial tree based on the NPI method for an American call option

with $i \in \{1, \ldots, T+1\}$ where S_T^i is the stock price in case i at the maturity T. Then rolling back to evaluate this call option for each node i from time $T-1$ to 0 on the basis of the definition $\overline{V_t^i} = \max\left\{S_t^i - K_c, (1+r)^{-1}[\overline{P_t^i}\ \overline{V_{t+1}^i} + (1-\overline{P_t^i})\overline{V_{t+1}^{i+1}}]\right\}$ for the upper value and $\underline{V_t^i} = \max\left\{S_t^i - K_c, (1+r)^{-1}[\underline{P_t^i}\ \underline{V_{t+1}^i} + (1-\underline{P_t^i})\underline{V_{t+1}^{i+1}}]\right\}$ for the lower value, where $\overline{P_t^i} = \overline{P}(Y_{n+t}^{n+t+1} = 1 | Y_i^n = s, Y_n^{n+t} = t+1-i) = \frac{s+t-i+2}{n+t+1}$ is the NPI upper probability for the node i at time t derived from Equation (5) and $\underline{P_t^i} = \underline{P}(Y_{n+t}^{n+t+1} = 1 | Y_i^n = s, Y_n^{n+t} = t+1-i) = \frac{s+t-i+1}{n+t+1}$ is the NPI lower probability for the node i at time t from Equation (6), S_t^i is the underlying asset for node i at time t. Here n is the number of historical data available and s is the number of upward movements in the historical data. Based on the general formula for the American option pricing, Equations (12) and (13), the formulae for each node in the binomial tree based on the backward NPI pricing method for an American call option are formulated as follows.

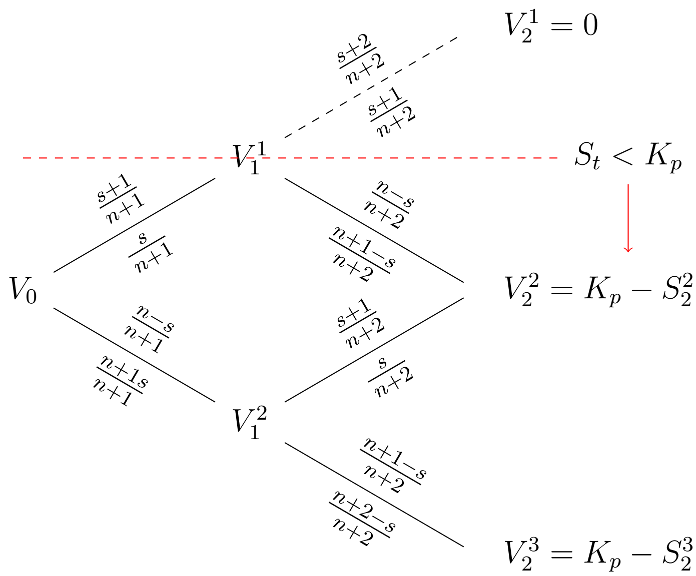


Figure 3: The binomial tree of the American put option (in the money)

The maximum buying price of an American call option

$$\begin{aligned}
\underline{V}^i_{t\{i=1\ldots t+1,\ t=0\ldots T-1\}} &= \max\left\{S_t^i - K_c, (1+r)^{-1}\left[\underline{P_t^i}\ \underline{V_{t+1}^i} + (1-\underline{P_t^i})\underline{V_{t+1}^{i+1}})\right]\right\} \\
&= \max\left\{S_t^i - K_c, (1+r)^{-1}\left[\frac{s+t-i+1}{n+t+1}\underline{V_{t+1}^i} + \frac{n-s+i}{n+t+1}\underline{V_{t+1}^{i+1}}\right]\right\} \\
\underline{V}^i_{T\{i=1\ldots T+1\}} &= \max\{0, S_T^i - K_c\}
\end{aligned} \tag{14}$$

The minimum selling price of an American call option

$$\begin{aligned}
\overline{V}^i_{t\{i=1\ldots t+1,\ t=0\ldots T-1\}} &= \max\left\{S_t^i - K_c, (1+r)^{-1}\left[\overline{P_t^i V_{t+1}^i} + (1-\overline{P_t^i})\overline{V_{t+1}^{i+1}}\right]\right\} \\
&= \max\left\{S_t^i - K_c, (1+r)^{-1}\left[\frac{s+t-i+2}{n+t+1}\overline{V_{t+1}^i} + \frac{n-s+i-1}{n+t+1}\overline{V_{t+1}^{i+1}}\right]\right\} \\
\overline{V}^i_{T\{i=1\ldots T+1\}} &= \max\{0, S_T^i - K_c\}
\end{aligned} \tag{15}$$

3.2 American put option

The binomial tree for the American put option is displayed in Figure 3. V_t^i with $t \in \{0, \ldots, T\}$ and $i \in \{1, \ldots, t+1\}$ is the put option value in node i at time t. Similar to the call option pricing procedure, we start to evaluate the put option at the maturity $V_T^i = \max\{0, K_p - S_T^i\}$ with $i \in \{1, \ldots, T+1\}$ and S_T^i is the stock price in node i at the maturity T. To evaluate this put option for each node i from time $T-1$ to 0, we use formulae $\overline{V_t^i} = \max\left\{K_p - S_t^i, (1+r)^{-1}[\underline{P_t^i}\overline{V_{t+1}^i} + (1-\underline{P_t^i})\overline{V_{t+1}^{i+1}}]\right\}$ for the upper value and $\underline{V_t^i} = \max\left\{K_p - S_t^i, (1+r)^{-1}[\overline{P_t^i}\underline{V_{t+1}^i} + (1-\overline{P_t^i})\underline{V_{t+1}^{i+1}}]\right\}$ for the lower value, which are derived from Equations (12) and (13), with NPI upper probability, $\overline{P_t^i} = \frac{s+t-i+2}{n+t+1}$ and with NPI lower probability, $\underline{P_t^i} = \frac{s+t-i+1}{n+t+1}$. This leads to the following results.

The maximum buying price of an American put option

$$\begin{aligned}\underline{V_t^i}_{\{i=1\ldots t+1\}} &= \max\left\{K_p - S_t^i, (1+r)^{-1}\left[\overline{P_t^i}\underline{V_{t+1}^i} + (1-\overline{P_t^i})\underline{V_{t+1}^{i+1}}\right]\right\} \\ &= \max\left\{K_p - S_t^i, (1+r)^{-1}\left[\frac{s+t-i+2}{n+t+1}\underline{V_{t+1}^i} + \frac{n-s+i-1}{n+t+1}\underline{V_{t+1}^{i+1}}\right]\right\} \qquad (16) \\ \underline{V_T^i}_{\{i=1\ldots T+1\}} &= \max\{0, K_p - S_T^i\}\end{aligned}$$

The minimum selling price of an American put option

$$\begin{aligned}\overline{V_t^i}_{\{i=1\ldots t+1\}} &= \max\left\{K_p - S_t^i, (1+r)^{-1}\left[\underline{P_t^i}\overline{V_{t+1}^i} + (1-\underline{P_t^i})\overline{V_{t+1}^{i+1}}\right]\right\} \\ &= \max\left\{K_p - S_t^i, (1+r)^{-1}\left[\frac{s+t-i+1}{n+t+1}\overline{V_{t+1}^i} + \frac{n-s+i}{n+t+1}\overline{V_{t+1}^{i+1}}\right]\right\} \qquad (17) \\ \overline{V_T^i}_{\{i=1\ldots T+1\}} &= \max\{0, K_p - S_T^i\}\end{aligned}$$

The outcome of the NPI method is an interval expected values, and the range of this interval depends on the relative number of historical data n. With a fixed m, the more historical data is available, the narrow the interval results.

4. EARLY EXERCISE OF AN AMERICAN OPTION

The rational trading theory proposed by Merton is commonly accepted in classic pricing methods. Merton (1973) showed that for an American call option without dividends the optimal stopping time is its expiry time, meaning that it is not optimal to exercise an American call option early. However, many scholars have found that the early exercise American call option with no dividend can happen in the real market. Zivney (1991) confirmed the early exercise of American call options without dividends by calculating the early exercise premium. Engstrom (2002) and Jensen and Pedersen(2006) found that in the empirical market there exist cases of early exercise American call options without dividends, especially when the market is frictional. This section will discuss the reason for this phenomenon and show that it does not hold for the NPI pricing method.

In the binomial tree, we used the backward method to calculate the American option price. At each node i at time t, the instant value $h_t(S_t = S_t^i)$ is compared to the discounted holding value $H_t(S_t = S_t^i)$, so the value of this node at time t is $V_t(S_t = S_t^i) = \max\{h_t(S_t = S_t^i), H_t(S_t = S_t^i)\}$. The holding value at time t is equal to the discounted expected value at time $t+1$, $H_t(S_t = S_t^i) = B(t, t+1)V_{t+1}(S_{t+1}|S_t = S_t^i)$. To start our study of the NPI method, an example is presented to aid understanding of the pricing method.

Example 4.1

In this example, there is an American call option with maturity $T = 2$, which is in the money (ITM), $(K_c < S_0)$. The binomial tree of the stock price and option value at each node $V_t(S_t = S_t^i)$ are listed in Figure 4 (in the parenthesis, maximum buying price on the left and minimum selling price on the right).

A stock with initial price $S_0 = 16$ will either move up by factor $u = 1.1$ or down by $d = 0.9$. As historical observations, there are 50 data available in the market, and among them 26 are up. Then NPI upper and lower probabilities of each movement are also displayed in Figure 4. We assume in this example that the constant discount rate r is equal to the lower expected return of the stock price during the time period from time 0 to time 1, $r = u\frac{s}{n} + d\frac{n-s+1}{n+1} - 1 \approx 0.002$. Based on the definition of the American call option, we want to see whether there is any possibility of early exercise for this call option. It turns out that the lowest branch of binomial tree from time 1 to time 2, highlighted in the square in Figure 4, is optimal to be exercised early. Because the lower discounted expected payoff at time 2, which is equal to $(1 + 0.002)^{-1}(3.8\frac{26}{52} + 0.2\frac{26}{52}) \approx 1.996$ less than 2, the instant payoff at time 1, while the upper discounted expected payoff at time 2, which is equal to

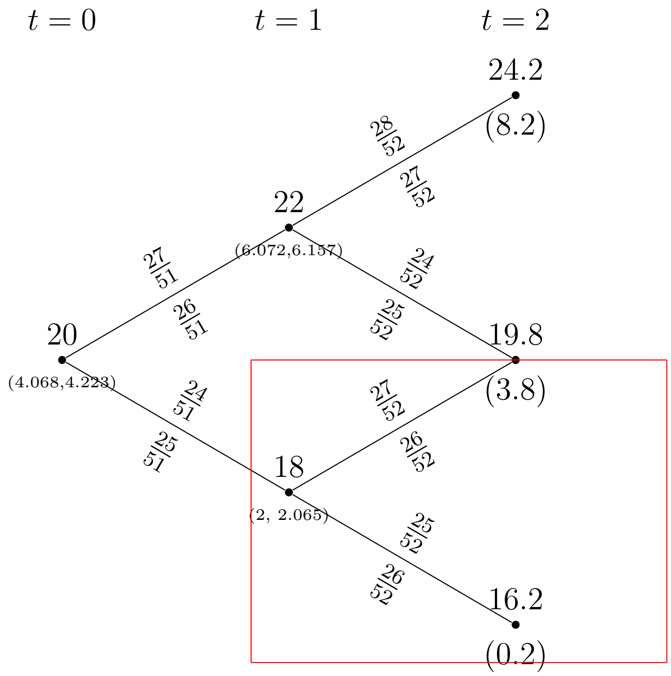


Figure 4: Example 4.1

$(1+0.002)^{-1}(3.8\frac{27}{52}+0.2\frac{25}{52}) \approx 2.065$ is greater than 2, the instant payoff at time 1. This means that, if the NPI investor is the option holder, he will exercise this option early, but he does not expect this option to be early exercised as an option writer. Therefore, for the same option, he is willing to sell at a price as an European call option but willing to buy it at a higher price than an European call option.

4.1 When to exercise an American call option early

From Example 4.1, we know that the result shows a sharp contrast to the rational trading theory. Since the NPI method does not assume the complete knowledge of the stochastic nature of the process like the classic method does but learns from the historical data, this leads to a substantially different theory, most noticeably resulting in the porssibility that it may be optimal to exercise an American non-dividend call option early. As for NPI method there are two bounds, upper and lower bounds, actually an investor's trading position will decide which bound he should focus on. As an option holder, the investor should focus on the maximum buying price, while an option seller is supposed to pay attention to the minimum selling price. Then to study the condition of early exercise, we could simplify the procedure by only focusing on either the maximum buying price or the minimum selling price according to two trading positions. Here we discuss to early exercise an American call option, then the NPI investor is supposed to hold an American call option and willing to buy the underlying asset in the future if there exist some profits. Because the lower expected return varies in every time step, we set the discount rate equal to constant r, then at time $t+1$ there will be two circumstances of the stock lower expected return $\underline{r_{t+1}}$ different from r, namely higher than r or lower than r, where $\underline{r_{t+1}} = u\underline{P_t}(S_t) + d(1 - \underline{P_t}(S_t)) - 1$ following $\underline{E}(S_{t+1})(1+\underline{r_{t+1}})^{-1} = S_t$. Therefore, the discounted expected stock price $\underline{E}(S_{t+1})(1+r)^{-1}$ at time $t+1$ is not always equal to the stock maximum buying price S_t at t, while in risk-neutral evaluation we always have $E(S_{t+1})(1+r_f)^{-1} = S_t$.

To get the early exercise condition for an American call option holder, we compare the option instant value $h_t(S_t = S_t^i) = S_t - K_c$ to the holding value $H_t(S_t = S_t^i) = \underline{E}[S_{t+1} - K_c]^+(1+r)^{-1}$ at time t for each node i. If $S_t - K_c > \underline{E}[S_{t+1} - K_c]^+(1+r)^{-1}$, it is best to exercise this call option, otherwise to hold it. Here $\underline{E}[S_{t+1} - K_c]^+(1+r)^{-1}$ is computed based on the option value of two nodes at time $t+1$, $V_{t+1}(S_{t+1} = S_t^i u)$ and $V_{t+1}(S_{t+1} = S_t^i d)$. Therefore, before the comparison we need to consider the exercise of two nodes at time $t+1 \neq T$, which consist of three circumstances: two nodes are exercised at time $t+1$, one is exercised while the other is better to be held, and two nodes are held. For the first circumstances, both nodes at time $t+1$ are exercised, the condition for early exercise an American call option is as follows.

$$
\begin{aligned}
&\underline{E}[S_{t+1} - K_c]^+(1+r)^{-1} \geq \underline{E}(S_{t+1} - K_c)](1+r)^{-1} > (S_t - K_c) \\
&\Leftrightarrow \underline{E}[S_{t+1}] - K_c > (S_t - K_c)(1+r) \\
&\Leftrightarrow S_t(1 + \underline{r_{t+1}}) - K_c > (S_t - K_c)(1+r) \\
&\Leftrightarrow \underline{r_{t+1}} > (1 - \frac{K_c}{S_t})r \qquad (18) \\
&\Leftrightarrow \underline{P_t}(S_t) > \frac{(1+r-d)S_t - rK_c}{(u-d)S_t} \qquad (19)
\end{aligned}
$$

Here, r is the non-negative discount rate, and $\underline{r_{t+1}}$ is lower expected return of the stock price at time $t+1$. We can express the condition for holding this call option at S_t not only as $\underline{r_{t+1}} > (1 - \frac{K_c}{S_t})r$ but also as a condition on the NPI lower probability of S_t moving upwards, $\underline{P_t}(S_t) > \frac{(1+r-d)S_t - rK_c}{(u-d)S_t}$. This is derived due to the relationship between the lower expected stock return and the lower probability, $1 + \underline{r_{t+1}} = u\underline{P_t}(S_t) + (1 - \underline{P_t}(S_t))d$.

For the circumstance that the option of one node at $t+1$ is optimal to be held while the other is exercised early, there exist two different situations, the first one is that the upward node is optimal to be held while the downward node is optimal to be exercised early. In this situation, the upward node contains the option value, which is the holding value at time $t+1$ represented as $H_{t+1}(S_t u)$, which is greater than the instant value at time $t+1$, $H_{t+1}(S_t u) > h_{t+1}(S_t u) \Leftrightarrow H_{t+1}(S_t u) > S_t u - K_c$. The downward node value is the instant value $h_t(S_t d) = [S_t d - K_c]^+$. The early exercise condition for this situation now becomes

$$
\begin{aligned}
&(1+r)^{-1}[\underline{P_t}(S_t)H_{1+t}(S_t u) + (1 - \underline{P_t}(S_t))[S_t d - K_c]^+] > (S_t - K_c) \\
&\Leftrightarrow (1+r)^{-1}[\underline{P_t}(S_t)(S_t u - K_c + a) + (1 - \underline{P_t}(S_t))(S_t d - K_c)] > (S_t - K_c) \\
&\Leftrightarrow \underline{r_{t+1}} > r(1 - \frac{K_c}{S_t}) - \frac{\underline{P_t}(S_t)a}{S_t} \qquad (20) \\
&\Leftrightarrow \underline{P_t}(S_t) > \frac{(1+r-d)S_t - rK_c}{(u-d)S_t + a} \qquad (21)
\end{aligned}
$$

where a is the difference between the discounted expected value and instant value at time $t+1$ for upward node, $a = H_{t+1}(S_t u) - h_{t+1}(S_t u) = H_{t+1}(S_t u) - (S_t u - K_c)$, depending on all future paths related to the node where $S_{t+1} = S_t u$ from time $t+1$ to maturity.

Another possible situation in this circumstance is that the upward node is optimal to be exercised but the downward node is optimal to be held. In order to find the condition of holding the option at time t, we compare the discounted expected value at time t, $H_t(S_t) = (1+r)^{-1}[\underline{P_t}(S_t)(S_t u - K_c) + (1 - \underline{P_t}(S_t))H_{t+1}(S_t d)]$, and the instant value at time t, $h(S_t) = S_t - K_c$. This leads to the early exercise condition

$$(1+r)^{-1}[\underline{P_t}(S_t)(S_tu-K_c)+(1-\underline{P_t}(S_t))H_{t+1}(S_td)]>(S_t-K_c)$$

$$\Leftrightarrow (1+r)^{-1}[\underline{P_t}(S_t)(S_tu-K_c)+(1-\underline{P_t}(S_t))(S_td-K_c+b)]>(S_t-K_c)$$

$$\Leftrightarrow \underline{r_{t+1}} > r(1-\frac{K_c}{S_t})-\frac{(1-\underline{P_t}(S_t))b}{S_t} \tag{22}$$

$$\Leftrightarrow (\underline{P_t}(S_t) > \frac{(1+r-d)S_t-rK_c-b}{(u-d)S_t-b}) \tag{23}$$

where $b=H_{t+1}(S_td)-h_{t+1}(S_td)=H_{t+1}(S_td)-(S_td-K_c)$ with $S_td-K_c<0$. Because if $S_tu-K_c>H_{t+1}(S_tu)$ then $S_td-K_c>H_{t+1}(S_td)$ unless $S_td-K_c<0$.

The last circumstance is clear, because both future nodes at time $t+1$ are optimal to be held, so the node at time t is optimal to be held as well. For circumstances investigated above, the American call option has a positive instant payoff at time t, otherwise the investor has to hold it for future time steps. We formulate this result as a theorem.

Theorem

If $\underline{r_{t+1}} > r$, then the American call option should be held.

If an American call option is exercised at time t, then $\underline{r_{t+1}} < r$.

Proof

As $(1-\frac{K_c}{S_t})r > r(1-\frac{K_c}{S_t})-\frac{\underline{P_t}(S_t)a}{S_t}$ and $(1-\frac{K_c}{S_t})r > r(1-\frac{K_c}{S_t})-\frac{(1-\underline{P_t}(S_t))b}{S_t}$, we could conclude that if $\underline{r_{t+1}} > (1-\frac{K_c}{S_t})r$, the call option should be held at time t. Moreover, the upper boundary of $(1-\frac{K_c}{S_t})r$ is r, then if $\underline{r_{t+1}} > r$, the American call option should be held. On the contract, if an American call option is exercised at time t, we know that $\underline{r_{t+1}}$ does not follow the holding condition, at least lower than the upper boundary of the holding condition r. Thus, if an American call option is exercised at time t, then $\underline{r_{t+1}} < r$. Note that these conditions in Theorem 4.1. are sufficient conditions, but not necessary conditions.

For the American call option selling position, replacing $\underline{r_{t+1}}$ in all conditions of different circumstances, Equations (18), (20) and (22), with $\overline{r_{t+1}}$ leads us to the early exercise conditions.

4.2 When to exercise an American put option early

Similarly, following the NPI pricing method for a non-dividends American put option, it is possible to gain more profit when it is exercised prematurely than at maturity. Here the NPI investor's position is buying a put option and willing to sell the underlying asset at a higher price in the future. So the lower NPI expected option value of the put option and the minimum selling stock price should be in this comparison. Same as for the call option, we only focus on the one step binomial tree instead of the whole tree, because NPI probabilities for each step in the tree change with the data.

For a put option, there are three circumstances for the option of two nodes after stock price movements in the one step tree, the upward node $V_{t+1}(S_{t+1} = S_t^i u)$ and the downward node $V_{t+1}(S_{t+1} = S_t^i d)$. It is possible that the option for both two nodes is optimal to be exercised early, or that one is exercised prematurely while the other one is better to be held, or that the option for both nodes are worth to be held. For the first circumstance, referring to the definition of the early exercise, as long as the discounted expected value at time t, $B(t, t+1)V_{t+1}(S_{t+1}|S_t) = \underline{E}[K_p - S_{t+1}]^+(1+r)^{-1}$, is greater than the instant value at time t, $h(S_t) = K_p - S_t$, then it is optimal to be held. Because the option holder is going to sell the stock at the minimum selling price when he exercises the option, $\underline{E}(K_p - S_{t+1}) = K_p - \underline{E}(S_{t+1}) = K_p - S_t(1 + \overline{r_{t+1}})$. Here $\overline{r_{t+1}}$ is related to the upper NPI probability, $\overline{r_{t+1}} = \overline{P_t}(S_t)u + (1 - \overline{P_t}(S_t))d - 1$. The condition for holding this option is,

$$
\begin{aligned}
&\underline{E}[K_p - S_{t+1}]^+(1+r)^{-1} \geq \underline{E}(K_p - S_{t+1})(1+r)^{-1} > K_p - S_t \\
&\Leftrightarrow K_P - S_t(1 + \overline{r_{t+1}}) > (K_p - S_t)(1+r) \\
&\Leftrightarrow \overline{r_{t+1}} < (1 - \frac{K_p}{S_t})r \qquad (24) \\
&\Leftrightarrow \overline{P_t}(S_t) < \frac{(1+r-d)S_t - rK_p}{(u-d)S_t} \qquad (25)
\end{aligned}
$$

Because $(1-\frac{K_P}{S_t})r < 0$, unless the stock has a very small expected return, in this circumstance the condition for holding a put option until the maturity is harder to achieve than holding a call option until the maturity.

For the second circumstance, for one node the option is exercised at time $t+1$ and for the other note is optimal to hold the option. Similar to the American call option, for this circumstance we have two different situations, the option of the upward node is exercised early and the downward one is not, or the other way around. For the first situation, the holding value at time t is $H_t(S_t) = (1+r)^{-1}[\overline{P_t}(S_t)[K_p - S_t u]^+ + (1-\overline{P_t}(S_t))H_{t+1}(S_t d)]$, where $[K_p - S_t u]^+$ is the instant value at the upward node, and $H_{t+1}(S_t d)$ is the holding value at the downward node.

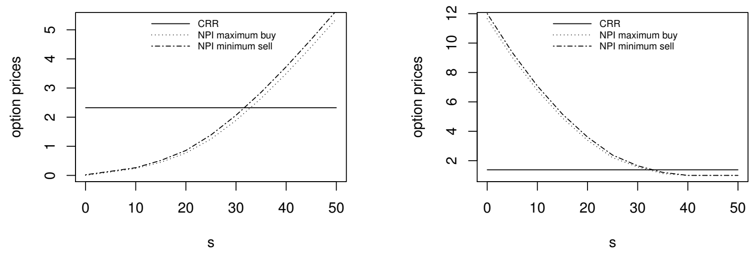
$$
\begin{aligned}
&(1+r)^{-1}[\overline{P_t}(S_t)[K_p - S_t u]^+ + (1-\overline{P_t}(S_t))H_{t+1}(S_t d)] \\
&\Leftrightarrow (1+r)^{-1}[\overline{P_t}(S_t)(K_p - S_t u) + (1-\overline{P_t}(S_t))(K_p - S_t d + v)] > K_p - S_t \\
&\Leftrightarrow \overline{r_{t+1}} < (1 - \frac{K_P}{S_t})r + \frac{(1-\overline{P_t}(S_t))v}{S_t} \qquad (26)\\
&\Leftrightarrow \overline{P_t}(S_t) < \frac{(1+r-d)S_t - rK_p + v}{(u-d)S_t + v} \qquad (27)
\end{aligned}
$$

Here $v = H_{t+1}(S_t d) - h_{t+1}(S_t d) = H_{t+1}(S_t d) - (K_p - S_t d)$. If the stock price S_t at time t is the same as in the first circumstance, then it is clear that this condition is easier to reach than that in the first circumstance. Another situation is that the option of the downward node is optimal to be exercised early, and the upward one is optimal to be held. The comparison between the holding value $H_t(S_t) = (1+r)^{-1}[\overline{P_t}(S_t)H_{t+1}(S_t u) + (1-\overline{P_t}(S_t))[K_p - S_t d]^+]$ and the instant value $K_p - S_t$ leads to

$$
\begin{aligned}
&(1+r)^{-1}[\overline{P_t}(S_t)H_{t+1}(S_t u) + (1-\overline{P_t}(S_t))[K_p - S_t d]^+] \\
&\Leftrightarrow (1+r)^{-1}[\overline{P_t}(S_t)(K_p - S_t u + w) + (1-\overline{P_t}(S_t))(K_p - S_t d)] > K_p - S_t \\
&\Leftrightarrow \overline{r_{t+1}} < (1 - \frac{K_P}{S_t})r + \frac{\overline{P_t}(S_t))w}{S_t} \qquad (28)\\
&\Leftrightarrow \overline{P_t}(S_t) < \frac{(1+r-d)S_t - rK_p}{(u-d)S_t - w} \qquad (29)
\end{aligned}
$$

where $w = H_{t+1}(S_t u) - h_{t+1}(S_t u) = H_{t+1}(S_t u) - (K_p - S_t u)$.

For the last circumstance, the options for two nodes at time $t+1$ are both optimal to be held, of course at time t we should not do anything towards this option. All these circumstances are settled based on the assumption that the instant value at time t is positive,



(a) Option prices with varying s for the call option

(b) Option prices with varying s for the put option

Figure 5: Comparison between the CRR model and the NPI method for American options

for those $K_p - S_t \leq 0$. For an American put option seller, the conditions could be formulated by replacing $\overline{r_{t+1}}$ with $\underline{r_{t+1}}$ in Equations (24), (26) and (28).

Theorem

If $\overline{r_{t+1}} < (1 - \frac{K_p}{S_t})r$, then the American put option should be held at time t.

Proof For an American put option, $(1 - \frac{K_p}{S_t})r$ is the lower boundary of all holding conditions, $(1 - \frac{K_p}{S_t})r < (1 - \frac{K_P}{S_t})r + \frac{(1-\overline{P}_t(S_t))v}{S_t}$ and $(1 - \frac{K_p}{S_t})r < (1 - \frac{K_P}{S_t})r + \frac{\overline{P_t}(S_t))w}{S_t}$ with constant positive values w and v. Thus, if the current upper expected return at time t is greater than $(1 - \frac{K_p}{S_t})r$, then it is optimal to hold this put option till the next time step. Note that the condition in Theorem 4.2. is a sufficient condition, but not a necessary one.

5. COMPARISON OF CRR AND NPI FOR AMERICAN OPTIONS

It is interesting to compare the CRR model and the NPI method for American option pricing. Following the procedure of the comparison for European options by He et al. (2019), the performance study is targeting on the profit and loss of an investor using the NPI method and trading with the only other investor using the CRR model in two scenarios, namely that the CRR investor right or wrong about the market assumptions.

We first plot the American option prices from the CRR model and the NPI model with fixed n but varying s. In Figure 5, we study the comparison based on the same underlying asset ($S_0 = 20$, $K = 21$, $u = 1.1$, $d = 0.9$, $q = 0.65$, $n = 50$, $m = 4$). The CRR interest rate is equal to 0.03, which is calculated from the CRR model as $r_{CRR} = qu + (1-q)d - 1$ Hull (2009). While for the NPI method the discount rate is calculated based on s and n, $r = \frac{s}{n}u + (1-\frac{s}{n})d - 1$. In Figure 5, for the call option when s varies from 0 to 50 the NPI maximum buying and minimum selling prices are getting greater, while the CRR price is a constant value intersecting with these two NPI prices. The two intersections are around value $nq = 32.5$. If s is lower than the left intersection then both NPI prices are lower than the CRR price, and if s is higher than the right intersection, both those prices are higher than the CRR price. If s is between the two intersections, the CRR price is between the NPI prices. For the put option, the pattern of the NPI prices are opposite to those in the call option graph. The maximum buying and minimum selling prices go down along with s increasing. There are also two intersections between the CRR price and two NPI prices around 32.5. When s is between the intersections, the CRR price is between the NPI prices. If s is lower than the left intersection the NPI prices are higher than the CRR price, while if s is higher than the right intersection the NPI prices are lower than the CRR price. The put option price approaches to 1 instead of 0 when s is close to n, for the optimal exercise time is zero with a positive payoff equal to 1 in this example.

5.1 Stopping times

Before the profit and loss calculation, it is necessary to study the different stopping times, the exercise times, of these two methods, as the profit and loss contains two parts. One is from the price and the other part is from the payoff. However, different exercise times give us different option payoffs, because the stock price at the exercise time changes with the time. We want to investigate the optimal stopping times of both the NPI and the CRR pricing methods. In order to do the comparison, there are some inputs needed in the R program: initial stock price S_0, upward movement factor u, downward movement factor d, predictive future time steps m, the constant probability q in the CRR model, strike price K, option type: call or put, and option trading position: buying or selling. The detailed steps for the simulation study in this paper performed in R are listed below:

1. Simulate N paths of stock price movements. To do that, the indicator of upward movement $I^i_{t\ (i\in\{1...N\},t\in\{1...m\})}$ is needed.

   – For the NPI method, based on the historical data, we generate the indicator of upward movement

   $$I^i_{t\ (i\in\{1...N\},t\in\{1...m\})} = \begin{cases} 1 & \text{upward movement} \\ 0 & \text{downward movement} \end{cases}$$

   $I^i_{t\ (i\in\{1...N\},t\in\{1...m\})} \sim Bin(1, p = \frac{s+\theta_t}{n+t})$ for buying the stock and $I^i_{t\ (i\in\{1...N\},t\in\{1...m\})} \sim Bin(1, p = \frac{s+\theta_t+1}{n+t})$ for selling the stock, where θ_t is the cumulated number of I^i_t from time zero to time t.

   – In terms of the CRR model $I^i_{t\ (i\in\{1...N\},t\in\{1...m\})} \sim Bin(1, q)$.

   – The stock price at each step is $S^i_t = S^i_{t-1} u^{I^i_t} d^{(1-I^i_t)}$ with $S^i_0 = S_0$.

2. Calculate the instant value of each step, $h_t = S^i_t - K_c$ for call option and $h_t = K_p - S^i_t$ for put option. For the CRR model there is only one stock price in this calculation. Because there are two prices generated from the NPI method, the stock price in this calculation is chosen according to the trading positions. $S^i_t = S^i_{t-1} u^{I^i_t} d^{(1-I^i_t)}$: for buying a call option and selling a put option $I^i_{t\ (i\in\{1...N\},t\in\{1...m\})} \sim Bin(1, p = \frac{s+\theta_t}{n+t})$, for selling a call option and buying a put option $I^i_{t\ (i\in\{1...N\},t\in\{1...m\})} \sim Bin(1, p = \frac{s+\theta_t+1}{n+t})$.

3. Calculate the expected holding value of the option at time t, which is the discounted expected option value at time $t+1$ based on the NPI backward pricing method for an American option. As the option value at $t+1$ is the maximum of the instant value h_{t+1} and the expected holding value H_{t+1}, the expected holding value at t is $H_t = B(t, t+1) \max(h_{t+1}, H_{t+1})$, where $B(t, t+1)$ is the discount factor from t to $t+1$.

4. Compare the instant value to the holding value from the initial time, and stop at the first time τ when the instant value is greater than the holding value, then τ is the optimal time for exercise.

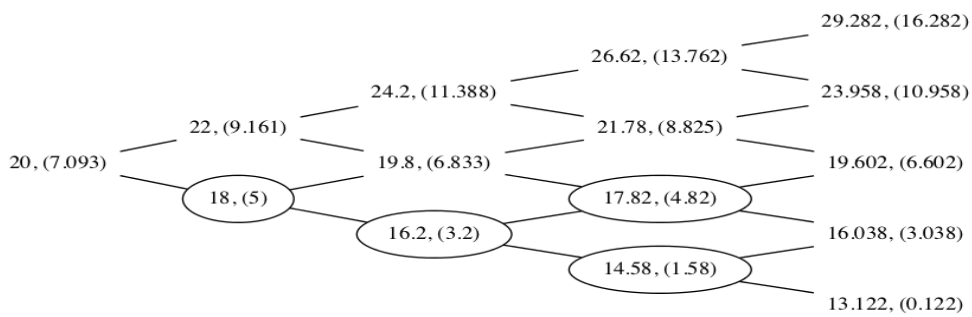


Figure 6: Tree plot for the NPI American call option

In this simulation, because we want to compare the stopping times between two methods, we simulate the stopping time of the same American option based on the same underlying asset under the same knowledge based on the CRR model and the NPI method. Because the information is the same towards two methods, so in the simulation for the NPI method $s \sim Bin(n, q)$. The discount rate is the expected stock return $r = r_{CRR}$. Here we try to explain the stopping time comparison between the CRR model and the NPI method in the light of examples. The first example is buying an in the money American call option with parameters as $K_c = 13$ and $T = 4$, on the basis of an underlying stock, $S_0 = 20$, $u = 1.1$, $d = 0.9$, $q = \frac{s}{n} = 0.52$, $s = 26$ and $n = 50$. As Theorem 4.1. says , $\underline{r_{t+1}} > r$ is a sufficient condition to hold the option. As the stopping time for the CRR model is always the same as the maturity, and we want to see the different case for the NPI method, thus, we pick this call option example that could lead us to an early exercise case.

After simulating $N = 20000$ paths for each pricing method, for the NPI method the optimal exercise time can be time 1, 3 or 4 depending on different paths of the underlying stock price, while for the CRR model, this call option is optimal to be exercised at the maturity time following the rational trading theory by Merton (1973). In the 20000 simulation routes of the NPI method, 9667 routes are stopped at time 1, 2396 routes at time 3, and 7937 routes at time 4. The average stopping time for the NPI method in this example is 2.40605. The reason that no route is stopped at time 2 can be explained from the binomial tree displayed in Figure 6 with the stock price and the option value in the parenthesis of

Table 1: Stopping time from both CRR and NPI method (20000 times simulation)

Method \ Time	0	1	2	3	4	5
NPI	0	0	3214	3877	0	12606
CRR	0	0	3214	3822	0	12964

each node. The node in a oval circle is the one optimal to be early exercised. As we can see from Figure 6, this American call option has a higher instant value than its corresponding holding at time 1, 2 and 3. However, if it attempts to reach early exercise node at time 2, it has already achieved the early exercise node at time 1. Then the investor will choose to exercise at time 1 rather than 2. Once again, this example proves that from the perspective of the NPI method, the rational trading theory is not true.

An example for a put option is also interesting. This time we choose to study the stopping time of selling an American put option. In order to make sure that the early exercise of the American put option happens, for this example we price a put option with the inputs $S_0 = K_p = 20$, $u = 1.1$, $d = 0.9$, $m = 5$, $n = 500$ and $q = \frac{s}{n} = 0.6$. From Table 1, we see that for both the CRR and NPI methods this American put option is optimal to be exercised before the maturity under some circumstances, and the stopping time is either 2, 3 or 5. However, The NPI method stops more often at time 3 and less often at time 5 than the CRR method, and both methods has the same times of stopping time 2 in this simulation. This is because the NPI method keeps learning from the data, then it assigns more probability to the path which has lower stock price with a higher payoff for selling the put option. Thus, when it comes to simulation the NPI method for selling the put option would have higher probability to encounter the early exercise case, and our results confirm this. The average stopping time of the CRR model is 4.1357, whereas the stopping time of the NPI method is 4.1302. This leading to a higher NPI option price 1.133 than the CRR option price 1.126.

These examples show that, there exists different stopping times between these two methods, especially for American call options. The different stopping times lead to a different payoff, and the expected option price is the expectation of payoffs after discounting. The stopping time differences between two methods actually play a very important role in the option price comparison. Therefore, this discussion of stopping time is really crucial to our study.

5.2 NPI profit and loss

In this section, we investigate the performance of the NPI method by calculating the profit and loss in a circumstance that an investor using the NPI method (the NPI investor) trades with an investor who uses the CRR method (the CRR investor). Inspired by the scenarios in European option study by He et al. (2019), we assume:

1. There are only two investors in the market, one uses the CRR model while the other one uses the NPI method.

2. The trade will be triggered when the CRR price is higher than or equal to the minimum selling price for the NPI method or lower than or equal to the maximum buying price for the NPI method, and the trading price will always be the NPI price, because we want to know the worse situation that the NPI investor will encounter.

We study the profit and loss (P&L) in two extreme scenarios: one is on the basis that the CRR model fits with the actual market, while the other is assumed that the CRR investor uses the wrong assumptions about the market. According to the paths of stock price simulated following the steps shown in Section 5.1, as well as the option prices from the two methods, we get the NPI profit and loss for each path based on different scenarios. Details of the calculations are presented below.

Scenario 1: CRR assumptions are correct

Assume that, in the market there is a real probability of upward movement p, and that the CRR assumed upward movement probability q is equal to p. When the NPI maximum buying price is higher than or equal to the CRR price, $\underline{V_0} \geq V_0^{CRR}$, the NPI investor will buy this American call or put option at this maximum buying price $\underline{V_0}$. At the stopping time, the NPI investor will get the option payoff calculated based on the CRR model, $V_\tau^{CRR} = \max_q \{0, S_T - K_c\}$ for call option, because in the CRR model an American call option will never be exercised early, and $V_\tau^{CRR} = \max_q \{0, K_p - S_\tau\}$ for put option. The exercise time follows $0 \leq \tau \leq T$. Due to different stopping times, a risk-free time value from τ to maturity T is used to calibrate the payoff, which means we assume that the payoff from an early exercise option will be invested in a risk-free product until the maturity. Then for a call option,

$$P\&L_c = V_\tau^{CRR}(1 + r_f)^{T-\tau} - \underline{V_0} = \max_q \{0, S_T - K_c\} - \underline{V_0} \tag{30}$$

and for a put option

$$P\&L_p = V_\tau^{CRR}(1 + r_f)^{T-\tau} - \underline{V_0} = \max_q \{0, K_p - S_\tau\} (1 + r_f)^{T-\tau} - \underline{V_0} \tag{31}$$

If the CRR price falls in the interval of NPI prices, $\underline{V_0} < V_0^{CRR} < \overline{V_0}$, then there is no trade. The NPI investor would like to sell the American option if he observes that the CRR price is higher than or equal to the minimum selling price $V_0^{CRR} \geq \overline{V_0}$. He will gain the option price $\overline{V_0}$ that will be invested in a risk free product under our assumptions. However, he may face a loss, $V_\tau^{CRR} = \max_q\{0, S_T - K_c\}$ for call option and $V_\tau^{CRR} = \max_q\{0, K_p - S_\tau\}$ for put option, when this option is exercised. The payoffs are simulated based on the CRR model with probability q, for the CRR in this scenario is always right. Then for a call option the profit and loss formula is,

$$P\&L_c = \overline{V_0}(1 + r_f)^T - V_\tau^{CRR} = \overline{V_0}(1 + r_f)^T - \max_q\{0, S_T - K_c\} \tag{32}$$

and for a put option

$$P\&L_p = \overline{V_0}(1 + r_f)^T - V_\tau^{CRR} = \overline{V_0}(1 + r_f)^T - \max_q\{0, K_p - S_\tau\} \tag{33}$$

Scenario 2: the CRR assumptions are wrong

We now consider the scenario in which the CRR assumptions are wrong, which means $q \neq p$. As an option buyer, the NPI investor will buy this option when the CRR price is lower than or equal to the maximum buying price, and exercise it at the optimal stopping time τ. However, as the CRR assumptions are wrong, meaning instead of the value q assumed by the CRR method for the probability of stock price upward movement, this probability is actually equal to p, the NPI investor will get the payoff as $V_\tau^p = \max_p\{0, S_T^p - K_c\}$ for the call option at the exercise time and $V_\tau^p = \max_p\{0, K_p - S_\tau^p\}$ for the put option. Here S_τ^p is simulated following the CRR simulation steps in Section 5.1, so actually V_τ^p is the option payoff calculated from the CRR model with a probability p instead of q, and we assume that this value as a real compensation of the option exercise from the market in this scenario. For time value calibration, we assume both the early exercise payoff and the earned option price will be invested in the risk free product until the maturity. So for a call option the NPI profit and loss formula is

$$P\&L_c = V_\tau^p(1+r_f)^{T-\tau} - \underline{V_0} = \max_p\{0, S_T^p - K_c\} - \underline{V_0} \tag{34}$$

and for a put option

$$P\&L_p = V_\tau^p(1+r_f)^{T-\tau} - \underline{V_0} = \max_p\{0, K_p - S_\tau^p\}(1+r_f)^{T-\tau} - \underline{V_0} \tag{35}$$

If $\underline{V_0} < V_0^{CRR} < \overline{V_0}$ then there is no transaction between the NPI investor and the CRR investor. When the CRR price is higher than or equal to the minimum selling price, $V_0^{CRR} \geq \overline{V_0}$, the NPI investor prefers to sell this option at the minimum selling price and save the money in a risk free account. When the CRR investor exercises this option, the NPI investor will pay the option payoff $V_\tau^p = \max_p\{0, S_T^p - K_c\}$ for the call option and $V_\tau^p = \max_p\{0, K_p - S_\tau^p\}$ for the put option. Then the NPI profit and loss could be formulated, for a call option

$$P\&L_c = \overline{V_0}(1+r_f)^T - V_\tau^p = \overline{V_0}(1+r_f)^T - \max_p\{0, S_T^p - K_c\} \tag{36}$$

and for a put option

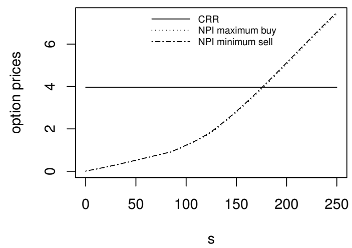
$$P\&L_p = \overline{V_0}(1+r_f)^T - V_\tau^p = \overline{V_0}(1+r_f)^T - \max_p\{0, K_p - S_\tau^p\} \tag{37}$$

Example 5.1

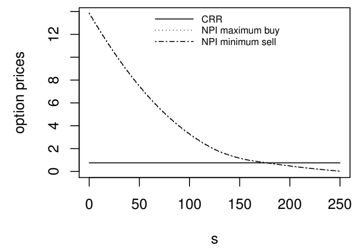
In this example, we calculate the profit or loss of the NPI investor when he is trading with the CRR investor in Scenario 1 and in Scenario 2. By investigating the NPI profit and loss in these two scenarios, we study the performance of the NPI method for American option pricing.

We randomize $\mathcal{S}$ according to two different scenarios. Two scenarios are distinguished by the different binomial distributions of s, in Scenario 1 $\mathcal{S} \sim Bin(n, q)$ whereas in Scenario 2 $\mathcal{S} \sim Bin(n, p)$. Based on $\mathcal{S}$, an average of profit and loss for the NPI investor is generated, which is the average value from N paths profit and loss. For our simulation, we randomly generate 1000 values for $\mathcal{S}_q$ or $\mathcal{S}_p$, and for each value $N = 10000$ stock price paths are simulated to be used as underlying asset prices. In our first example, Scenario 1 follows $\mathcal{S}_q \sim Bin(252, 0.7)$ with $q = p = 0.7$, and Scenario 2 follows $\mathcal{S}_p \sim Bin(252, 0.8)$ with $q = 0.7$ but $p = 0.8$. The underlying asset is still the same asset as in Example 4.1, with $S_0 = 20$, $u = 1.1$ and $d = 0.9$. We investigate at the money (ATM) options, meaning $K = S_0 = 20$. The option is an American option with $T = 5$ and discount rate $r = qu + (1 - q)d - 1$ for the CRR price and $r = \frac{s}{n}u + (1 - \frac{s}{n})d - 1$ for NPI prices. In this example we assume the interest rate for investment of the income of NPI investor before the maturity in the risk free market to be 0.002. For an NPI investor, buying a call option and selling the put option in this example is the wise choice, because the expectation of the stock price is positive leading to a positive expected payoff for the call option and zero expected payoff for the put option. This does not mean other trading positions would not give the investor a positive payoff, but these two trading actions have higher chances to encounter a profit than other actions. We calculate the average $P\&L$ for each $\mathcal{S}_q$ or $\mathcal{S}_p$, all outcomes are demonstrated in Figures 7 and 8.

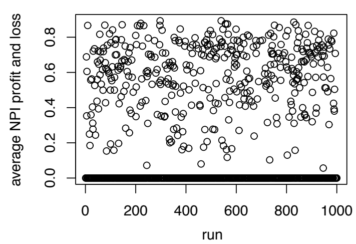
In Scenario 1, as we can see from in Figure 7, the NPI investor is involved in all four trading actions. Let us look at the call option first. Because $p = q = 0.7$, the expectation of $\mathcal{S}_q$ is 176. As long as $\mathcal{S}_q$ makes the maximum buying price higher than the CRR price, the trade of buying a call option is triggered. As acknowledged, the CRR call option will never



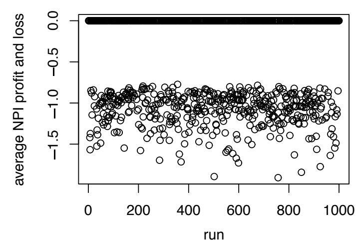
(a) NPI and CRR option prices (call)



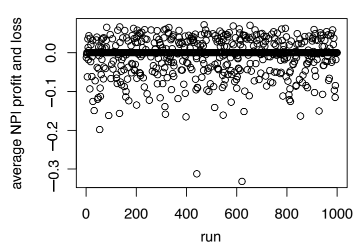
(b) NPI and CRR option prices (put)



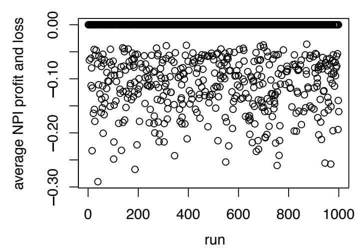
(c) NPI P&L for buying call option



(d) NPI P&L for selling call option



(e) NPI P&L for buying put option



(f) NPI P&L for selling put option

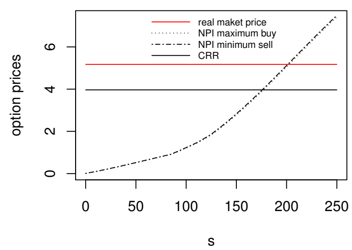
Figure 7: NPI profit and loss of Example 5.1 in Scenario 1 ($q = 0.7$, $n = 252$, $p = 0.7$)

be early exercised, so the NPI investor pays a higher or equal price and gets the CRR payoff at the maturity. In the Figure 7(c), it seems that the NPI investor earns some profit from buying the call option, meaning the payoff earned at the exercise time can offset the price payment. But due to the large number of n in the simulation, this result does not show all P&L outcomes. It is possible to encounter a negative value of profit and loss when $\mathcal{S}_q$ is very large making the maximum buying price so overvalued that the payoff can not offset it, even though this situation is rare.

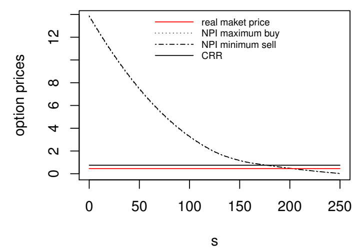
For selling call options, when $\mathcal{S}_q$ makes the minimum buying price lower than the CRR price, and it is higher than 126, making sure a non-negative discount rate for the NPI method, there is a trade of selling the call option. The NPI investor sells a call option at a lower or equal price to the CRR price saving this income into his bank account until the maturity, and pays to the CRR investor if the call option is exercised at the maturity. From Figure 7(d), it is obvious that the average NPI $P\&L$ for $\mathcal{S}_p$ is negative or zero, and there is a gap between the loss and zero. To explain this gap, we need to disclose the trading procedure of the NPI investor trading with the CRR investor at a CRR price, for all other cases lead to a greater loss. In this trade, with $\overline{V_0} = V_0^{CRR}$, the NPI investor sells this call option and puts the money $\overline{V_0} = V_0^{CRR}$ into his bank account, and he gets total profit from it as $V_0^{CRR} \times 1.002^5 = 4.0038$. However, the NPI investor needs to pay the CRR investor at the maturity for the exercised call option payoff $V_0^{CRR} \times 1.04^5 = 4.8228$, where the discount rate is $r = 0.7 \times 1.1 + 0.3 \times 0.9 - 1 = 0.04$. Then the minimum loss for the NPI call option seller is $L = V_0^{CRR} \times 1.04^5 - V_0^{CRR} \times 1.002^5 = 0.8190$. So if the NPI investor is selling the call option, his minimum loss is 0.8190.

If the NPI investor wants to buy a put option, he would face a profit or a loss. As long as $\mathcal{S}_q$ lower than or equal to the intersection between the maximum buying price and the CRR price, and it is higher than 126, the NPI investor buys the put option. If the put option is bought at a highest price when $s = 126$, and the put option buyer has a lower chance to get the payoff from this option, the average payoff earned at the exercise time is not great enough to offset the price payment. As $\mathcal{S}_q$ increases, the buying price gets lower. When the payoff to the CRR investor is higher than the price payment, the NPI investor has a positive profit. Figure 7(f) looks similar to Figure 7(d), with a smaller variety due to the lower CRR price and NPI minimum selling price. When $\mathcal{S}_q$ leads to a lower or equal minimum selling price than the CRR price, the NPI investor sells the put option and saves the money in his bank account, and at the exercise time he is possible to pay the payoff to the CRR buyer, when this put option is exercised. In expectation, as the stock price is expected to raise, the possibility of put option gets lower. However, because the CRR investor is using the true probability $p = q$, the NPI investor sells the put option at an undervalued price and has to face the payment in some cases.

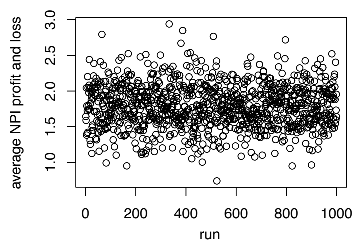
In Scenario 2, we consider the values $p = 0.8$, $q = 0.7$, so we simulate $\mathcal{S}_p \sim Bin(252, 0.8)$ with the expectation $E(\mathcal{S}_p) = 252 \times 0.8 = 202$. According to the P&L graphs in Figure 8, in this scenario the NPI investor only plays a role in buying the call option and selling the put option, which are the two wise and safe trading positions. For buying the call option, although the NPI investor is paying a higher price than in Scenario 1, he could also get a higher payoff at the maturity leading to a positive and higher profit than in Scenario 1. However, from Figure 8(d) of selling a call option, we could see that there is no profit or loss, because in this simulation we randomize 1000 $\mathcal{S}_p \sim Bin(252, 0.8)$. The expectation of $\mathcal{S}_p$ is 202, and around this value that all actions of buying the call option occur. As n is large enough to make sure $\mathcal{S}_p$ never leads to a minimum selling price which is lower than the CRR price, selling the call option is never going to happen in this simulation. It dose not mean that selling a call option can never take place in Scenario 2. In general, if n is not large enough, it could be the case, but it is still a very rare circumstance. We discuss this situation further in the next example. In our simulation, we did not observe the case of buying a put option in the simulations, which is shown as Figure 8(e), because in this simulation n is large enough. All $\mathcal{S}_p$ for selling a put option cause a successful action with either a profit or loss shown in Figure 8(f). The real market will have lower payoff for the put option, while the CRR model overvalues it. As the minimum selling price is close to the CRR price, even though the NPI investor sells the put option at a lower or equal price, the



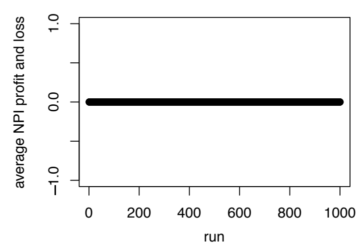
(a) NPI and CRR option prices (call)



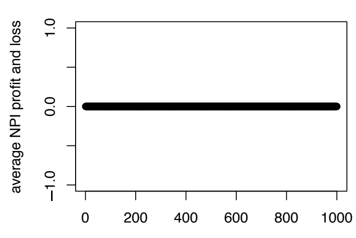
(b) NPI and CRR option prices (put)



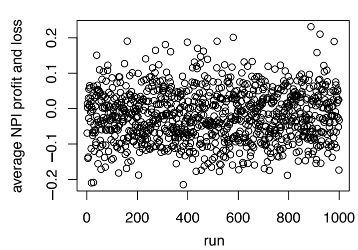
(c) NPI P&L for buying call option



(d) NPI P&L for selling call option

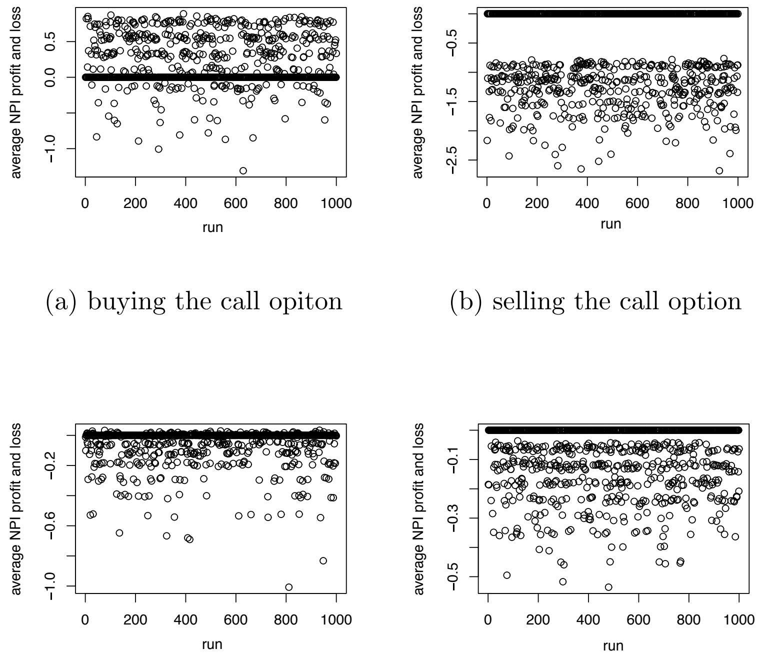


(e) NPI P&L for buying put option



(f) NPI P&L for selling put option

Figure 8: NPI profit and loss for Example 5.1 in Scenario 2 ($q = 0.7$, $n = 252$, $p = 0.8$)



(a) buying the call opiton (b) selling the call option

(c) buying the put option (d) selling the put option

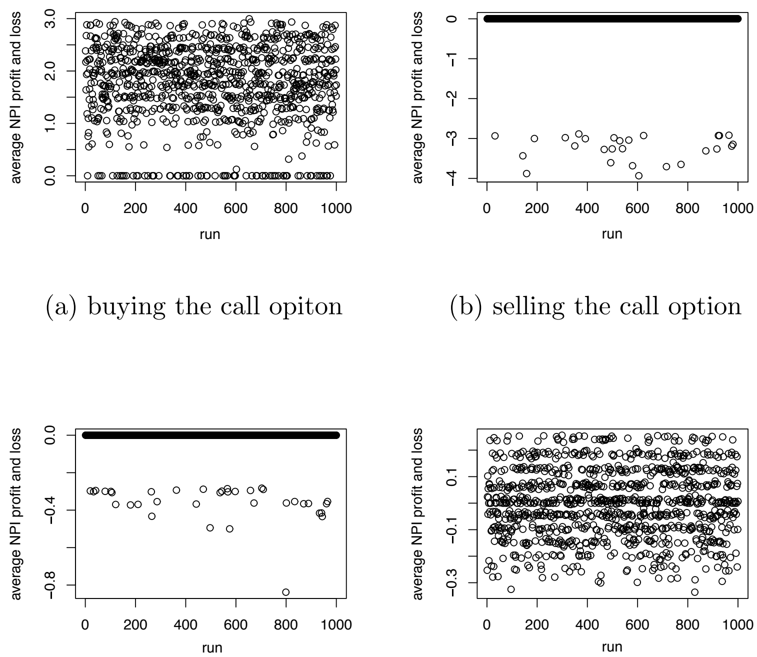
Figure 9: NPI profit and loss for Example 5.2 in Scenario 1 ($q = 0.7$, $n = 50$, $p = 0.7$)

payoff he eventually pays is lower than the CRR expectation, leading to a profit. As s gets larger, the minimum selling getting smaller, the option price gained from selling the option and its interests from the risk-free investment could not cover the payoff at the exercise time, then there exists a loss, but still the loss is lower than what happened in Scenario 1.

All in all, the NPI method performance is better in Scenario 2 than in Scenario 1 on the basis of this simulation. First it keeps the NPI investor away from the less safe trading action, selling the call option and buying the put option. And according to the P&L result of buying the call option and selling the put option, the NPI investor can make more profit and less loss in Scenario 2.

Example 5.2

To study the influence of the historical data size, we performed another simulation with



(a) buying the call opiton (b) selling the call option

(c) buying the put option (d) selling the put option

Figure 10: NPI profit and loss for Example 5.2 in Scenario 2 ($q = 0.7$, $n = 50$, $p = 0.8$)

historical data $n = 50$, the $\mathcal{S}_p \sim Bin(50, p)$, the results are displayed in Figures 9 and 10. Generally, the average NPI profit and loss is the same as that in Example 5.1, the NPI method would perform better than the CRR model in Scenario 2. However, smaller n incurs more loss to the NPI investor. In Scenario 1, as a call option buyer, the NPI investor can face some loss. In Figure 9(a), other than profit and no trade, there exists some loss as which did not occur in Figure 7(c). This phenomenon confirms what we discussed in Example 5.1. Because n is small, it is possible to reach the situation that the NPI investor pays a price higher than the payoff he would get at the maturity. For the same reason, when s is small the NPI investor could sell the call option at a very low price, then it eventually leads to greater loss to the NPI investor as shown in Figure 9(b). For the put option, because of a smaller n, no matter as a buyer or seller, the NPI investor could encounter an even worse case than that with a larger n.

The same situation also happens to the trade in Scenario 2. For the buying call option, the NPI investor will encounter the situation that the maximum buying price is lower than the CRR price making the NPI investor does not trade. As the case shown in the Figure 10(a), instead of all cases ending up with a positive payoff, as in Figure 8(c), here some cases leading no profit and loss meaning there is no trade. On the contrary, here it is possible to sell the call option causing a loss as shown in Figure 10(b). It is shown in Figure 10(c), a smaller historical data set could also expose the NPI put option buyer to a loss, as he may buy it at a high price that could not be compensated by the payoff. It is also easier in this case to sell the put option at a lower price which could lead to a more loss to the NPI investor than the loss that would occur with a larger n. Therefore, the large historical data is very important. In order to be sure that $\frac{s}{n}$ does not deviate from the real probability in the market a lot, n should be large enough. Then the prediction from the NPI method is more accurate to guide the investor to a right trading decision with more profit and less loss. Thus, how much the historical data should be in order to make the $\frac{\mathcal{S}_p}{n}$ is equal to p is another interesting question for future study.

Example 5.3

We also studied the P&L of the NPI investor when trading with the CRR investor in Scenario 2 in two other cases, namely with $q = 0.52$ and $q = 0.6$ while keeping $p = 0.8$ and $n = 50$, and the other inputs also the same as in Examples 5.1 and 5.2. In the case with $q = 0.6$, unlike Example 5.2, the NPI investor only participants in the trade of buying the call option and selling the put option, but not in selling the call option or buying the put option. The profit from the trade is slightly higher than the profit when $q = 0.7$, and the loss is slightly lower. When $q = 0.52$, the NPI investor still only does the wise trade. With a larger difference between p and q, the profit earned by the NPI investor is greater in the case with $q = 0.52$ than in which with $q = 0.6$, and for both these cases the profit of the NPI investor is greater than with $q = 0.7$ as studied in Example 5.2. This suggest that the NPI investor gets better payoff for large difference between q and p, which is in line with intuition. The figures and more detailed results will be presented in the first author's PhD thesis by He (2019).

From Examples 5.1, 5.2 and 5.3, when n is small, a large difference between q and p benefits the NPI method performance. The benefit reflects in two parts, the first one is that a large difference between q and p prevents the NPI investor getting involved in the unwise the trading action, selling the call option and buying the put option in our examples, and also leads more NPI investor's participance in the wise trading actions. Another part of benefit is a large difference between q and p helps the NPI investor earn more and loss less in the trade. This is because when n is small, $\frac{S_p}{n}$ can be far from p resulting in the NPI investor gets involved in a opposite trading position to the actual market. A large difference between q and p can reduce the chance of this situation happening, for the trading position of the NPI investor is affected by the CRR prediction which is leaded by q value. As the NPI investor behaves better in the trade, of course the P&L gets better along with it.

6. CONCLUDING REMARKS

We introduced the NPI method for American option pricing based on the backward solution method for the binomial tree model. NPI is an imprecise statistical method continuously learning from data. This property makes the NPI method for American option pricing more close to reality than the CRR model, which assumes that all information in the real world are fully known. With the NPI method for American option pricing, we could encounter the situation that it is optimal to early exercise an American call option with no dividends, which is a sharp contrast to the rational trading theory but happens in the real market. Therefore, the American option pricing procedure can not be described as a closed formula like European option pricing procedure does. The conditions to justify whether early exercise or holding further for both call and put options are listed in the paper. We also studied the average stopping time and option prices comparison between the CRR model and the NPI method by simulation, and we found that the stopping time of American options are different between the CRR model and the NPI method. Then we illustrated that NPI investor's profit and loss when trading with an investor who uses the CRR model in two scenarios. In Scenario 1 the CRR investor uses the right assumptions about the future market, and Scenario 2 is under the assumptions that the CRR investor does not use the correct assumption. The outcomes of some examples show the NPI investor gets a better payment in Scenario 2 than Scenario 1. This positive conclusion embodies in two parts, if historical data is large to make $\frac{S_p}{n}$ closed to p, one that the NPI investor only plays parts in the safer and wiser trade position, the other one that the P&L of the NPI investor is also more optimistic and that in Scenario 1 by earning more profit and lose less.

An interesting topic for future research is to investigate how much historical data could lead us to an accurate prediction. According to the example studied, it is better to include more historical data if one can safely assume that the future observations will be exchangeable with all the past data.

This paper presented the basic binomial tree model as a simple start in this research area, which is only used for ideal model situations. However, it helps us to understand a range of more realistic models for which we aim to investigate NPI in the future. In this paper we only used the comparison between two traders in the market, the NPI investor and the CRR investor, the results either with perfect or imperfect knowledge. The investigation of the NPI method application for American option pricing in real markets is another challenging future topic. This paper is an initial study, which more work is needed to develop the NPI approach for substantial real world applications in finance, e.g. when there are patterns in the historical data that may suggest the need of methods from time series. It may also be possible to improve the method by creating hybrid strategies based on multiple pricing methods and with multiple traders doing different scenarios. It is also interesting to develop the NPI method to more complicated options like exotic options.

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