Journal of Applied Statistics
Vol. 00, No. 00, February 2014, 1–22

RESEARCH ARTICLE

A New Weighted Rank Coefficient of Concordance

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(*v4.4 released October 2008*)

There are many situations where n objects are ranked by $b > 2$ independent sources or observers and in which the interest is focused on agreement on the top rankings. Kendall's coefficient of concordance [9] assigns equal weights to all rankings. In this paper, a new coefficient of concordance is introduced which is more sensitive to agreement on the top rankings. The limiting distribution of the new concordance coefficient under the null hypothesis of no association among the rankings is presented, and a summary of the exact and approximate quantiles for this coefficient is provided. A simulation study is carried out to compare the performance of Kendall's, the top-down and the new concordance coefficients in detecting the agreement on the top rankings. Finally, examples are given for illustration purposes, including a real data set from financial market indices.

Keywords: Coefficient of concordance; measures of agreement; measures of association; rankings; top-down coefficient of concordance; weights.

1. Introduction

There are many situations where n objects are ranked by two or more independent sources or observers and the interest is in measuring the agreement among these sets of rankings. Measures of agreement or association have large interest in practice, for example to evaluate the agreement between several experts, methods or models [1, 5, 12], to evaluate reproducibility [11], to search for species associations in community ecology [10], or to evaluate the agreement between the observed stock index future market prices and its theoretical prices in finance [15].

However, in many cases the interest is focused more on agreement on the top rankings than on the bottom rankings, as in sensitivity analysis where the aim is to determine the most influential variables [6]. Statistics such as Spearman's coefficient (for two sets of rankings) or Kendall's coefficient of concordance [8, 9] (for $b > 2$ sets of rankings) are not appropriate for such a scenario since they assign equal weights to all rankings. Iman and Conover [6] proposed the top-down coefficient of concordance (for $b \geq 2$ sets of rankings) based on Savage scores which is more sensitive to the agreement on the top rankings. Teles [18] carried out a simulation study to compare the performance of Kendall's and top-down concordance coefficients. In their study the top-down concordance coefficient was shown to perform better in detecting the agreement among the top or lower rankings compared to Kendall's coefficient. Maturi and Abdelfattah [13] presented a weighted rank correlation coefficient for two rankings. This paper presents a generalization of Maturi and Abdelfattah's weighted rank correlation coefficient for $b > 2$ which is

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ISSN: 0266-4763 print/ISSN 1360-0532 online

DOI: 10.1080/02664763.YYYY.XXXXXX
http://www.tandfonline.com

Table 1. Data structure

	Objects						
Observers	1	2	...	i	...	n	Total
1	R_{11}	R_{12}	...	R_{1i}	...	R_{1n}	$R_{1\cdot} = n(n+1)/2$
2	R_{21}	R_{22}	...	R_{2i}	...	R_{2n}	$R_{2\cdot} = n(n+1)/2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j	R_{j1}	R_{j2}	...	R_{ji}	...	R_{jn}	$R_{j\cdot} = n(n+1)/2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
b	R_{b1}	R_{b2}	...	R_{bi}	...	R_{bn}	$R_{b\cdot} = n(n+1)/2$
Total	$R_{\cdot 1}$	$R_{\cdot 2}$	...	$R_{\cdot i}$	...	$R_{\cdot n}$	$R_{\cdot\cdot} = bn(n+1)/2$

also more sensitive to the agreement on the top rankings and has the flexibility of choosing the weights that reflect the focus on the top rankings. We first introduce notation and provide a brief overview of some coefficients of concordance.

Suppose there are n objects which are all ranked by $b > 2$ independent sources or observers. This data structure can be visualized as a two-way layout table with b rows representing the observers and n columns representing the objects, see Table 1. Let R_{ji} denote the rank given by the jth observer to the ith object, the set of ranks in any row is a permutation of the numbers $1, 2, \ldots, n$ which sum up to $n(n+1)/2$. Let $R_{\cdot i} = \sum_{j=1}^{b} R_{ji}$, $i = 1, \ldots, n$, be the sum of the ranks assigned to the ith object taken over all b sets of rankings. The ranks in column i reflect the agreement between observers for object i. So if all ranks in column i are identical then this means that there is an agreement between all observers on object i, if this is the case for all objects then we can say that there is perfect agreement between all observers.

One may wish to test the null hypothesis of no agreement between the b rankings against the alternative of the existence of such agreement. To this end, there are several measures of association, called Concordance Coefficients, for such data. Perhaps the most obvious procedure is to average all values of Spearman's rank correlation coefficient, r_s, or Kendall's correlation coefficient, t_k, for the $\binom{b}{2}$ possible pairs, but this is evidently very tedious when b is large [8]. Below we briefly discuss two concordance coefficients that measure the overall agreement between the $b > 2$ rankings.

Kendall's coefficient of concordance, K, was introduced independently by Kendall and Smith [9] and Wallis [19], and is given by

$$K = \frac{12}{b^2 n(n^2-1)} \sum_{i=1}^{n} \left(R_{\cdot i} - \frac{b(n+1)}{2} \right)^2 \tag{1}$$

Kendall and Smith [9] showed that there is a relationship between K and the average value of the Spearman's rank correlation coefficient for the $\binom{b}{2}$ possible pairs, $\overline{r}_s$, where

$$\overline{r}_s = \frac{bK-1}{b-1}$$

The values of K range between 0 and 1, where 1 indicates that there is perfect agreement and 0 indicates that there is no agreement. Under the null hypothesis, the statistic $b(n-1)K$ has asymptotically (as $b \to \infty$) a chi-squared distribution with $n-1$ degrees of freedom [8]. Raghavachari [17] generalized Kendall's coefficient of concordance K to interval scaled data which is equal to K for rank order data. He also introduced a second measure of concordance which is based on the average value of Kendall's correlation coefficient $\bar{t}_k$ between the $\binom{b}{2}$ possible pairs.

Iman and Conover [6] proposed the top-down coefficient of concordance, T, which is more sensitive to the agreement on the top rankings,

$$T = \frac{1}{b^2(n - S_1)} \left(\sum_{i=1}^{n} S_{.i}^2 - nb^2 \right) \tag{2}$$

where $S_1 = \sum_{l=1}^{n} \frac{1}{l}$ and $S_{.i}$ is the sum of the Savage scores assigned to the ith object taken over all b sets of rankings. Savage scores are given by

$$S_{ji} = \sum_{l=i}^{n} \frac{1}{l} \quad \text{and} \quad S_{.i} = \sum_{j=1}^{b} S_{ji}$$

Similar to Kendall's coefficient of concordance K, the statistic $b(n-1)T$ has asymptotically (as $b \to \infty$) a chi-squared distribution with $n-1$ degrees of freedom [6].

In this paper, we introduce a new rank coefficient of concordance which is also more sensitive to the agreement on the top rankings and offers flexibility of choosing the weights that reflect the emphasis or focus on the top rankings. The new rank coefficient of concordance can be considered as a generalization of the weighted rank correlation coefficient for two rankings ($b = 2$) introduced by Maturi and Abdelfattah [13], which is reviewed in Section 2. The new coefficient of concordance is introduced in Section 3, and the limiting distribution of the new concordance coefficient under the null hypothesis of no association among the rankings is presented in Section 4. A summary of the exact and approximate quantiles for this coefficient is also provided in this section. A simulation study has been performed in order to investigate the performance of this new concordance coefficient compared to the two alternative concordance coefficients, the results are presented in Section 5. In order to illustrate the important features of the new rank coefficient of concordance, three examples are given in Section 6, including a real data set from financial market indices. Finally some concluding remarks are given in Section 7, and the proofs of the main results of the paper are presented in the appendix.

2. A weighted rank correlation coefficient for two rankings

Maturi and Abdelfattah [13] introduced a weighted rank correlation coefficient, R_w, to test the null hypothesis that two variables or two rankings are independent (so $b = 2$). This weighted rank correlation R_w is more sensitive to agreement on the top ranks, it is based on the weighted scores $(w^{R_{1i}}, w^{R_{2i}})$ where (R_{1i}, R_{2i}) are the paired rankings of object $i = 1, 2, \ldots, n$, and the weight w is any number in (0,1). Throughout this paper, we assume that there are no ties among rankings. Figure 1 shows the weighted ranks for $n = 5, 10, 25, 50$ and for $w = 0.1(0.1)0.9$, where w^r is the weight for the rank r, $r = 1, \ldots, n$ (i.e. $w^1, \ldots, w^n$). We can see the speed of decrease of the values w^r for decreasing weights from $w = 0.9$ to $w = 0.1$. In fact this choice of weights is attractive as the proportional difference between weights of consecutive ranks is constant, $(w^r - w^{r+1})/w^r = 1 - w$. Essentially, this is the same as using the exponential utility (or loss) function in decision theory, which models constant absolute risk aversion and has many applications in finance and other areas of decision support [2, 16]. It has the important practical advantage of being relatively easy to assess based on information from analysts, as it only

Table 2. Data structure with the weighted scores

	Objects						
Observers	1	2	...	i	...	n	Total
1	$w^{R_{11}}$	$w^{R_{12}}$	...	$w^{R_{1i}}$	...	$w^{R_{1n}}$	$\sqrt{a_1}$
2	$w^{R_{21}}$	$w^{R_{22}}$	...	$w^{R_{2i}}$	...	$w^{R_{2n}}$	$\sqrt{a_1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j	$w^{R_{j1}}$	$w^{R_{j2}}$	...	$w^{R_{ji}}$	...	$w^{R_{jn}}$	$\sqrt{a_1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
b	$w^{R_{b1}}$	$w^{R_{b2}}$	...	$w^{R_{bi}}$	...	$w^{R_{bn}}$	$\sqrt{a_1}$
Total	$w_{\cdot 1}$	$w_{\cdot 2}$	...	$w_{\cdot i}$	...	$w_{\cdot n}$	$b\sqrt{a_1}$

requires one parameter (w) to be chosen for which one can explicitly focus on the relevance of any two neighbouring ranks.

The weighted rank coefficient R_w is given by [13]

$$R_w = \left(n \sum_{i=1}^{n} w^{R_{1i}+R_{2i}} - a_1 \right) / (na_2 - a_1) \tag{3}$$

where $a_1 = w^2(1-w^n)^2/(1-w)^2$ and $a_2 = w^2(1-w^{2n})/(1-w^2)$. R_w in (3) can also be written as

$$R_w = \sum_{i=1}^{n} \left(\frac{w^{R_{1i}} - n^{-1}\sqrt{a_1}}{[a_2 - (a_1/n)]^{1/2}} \right) \left(\frac{w^{R_{2i}} - n^{-1}\sqrt{a_1}}{[a_2 - (a_1/n)]^{1/2}} \right) \tag{4}$$

The statistic R_w has a maximum value of 1, yet its minimum possible value is not -1. In fact, the minimum value of R_w is -1 only for $n = 2$ and increases away from -1 towards approximately from -2×10^{-6} to -3×10^{-4}, depending on the value of w. This is very similar behavior to the top-down correlation coefficient for $b = 2$ introduced by Iman and Conover [6]. Maturi and Abdelfattah [13] showed that R_w is a locally most powerful rank test. For $n \to \infty$ and under the null hypothesis of independence, they showed that the statistic $(n-1)^{1/2}R_w$ has asymptotically a standard normal distribution. Thus the critical region of size α for testing the null hypothesis of independence (against the alternative of positive correlation) is, for large n, given approximately by $R_w > z_{1-\alpha}/\sqrt{n-1}$ where $z_{1-\alpha}$ is the $1-\alpha$ quantile of the standard normal distribution. A summary of quantiles of the exact null distribution for $n = 3(1)9$ is provided by Maturi and Abdelfattah [13].

3. A new weighted rank concordance coefficient

We will use the weighted scores proposed by Maturi and Abdelfattah [13] to derive the new coefficient of concordance, and by using the notation introduced above, the weighted scores are $w^{R_{ji}}$ where R_{ji} is the rank given by the jth observer to the ith object and $0 < w < 1$. The choice of w reflects the desire to emphasize the top rankings. The data structure with these weighted scores is presented in Table 2. Of course the sum of each row is constant and equal to $\sqrt{a_1} = \sum_{i=1}^{n} w^{R_{ji}} = w(1-w^n)/(1-w)$, and the total average of all weighted scores in Table 2 is $\bar{w} = (1/bn)\sum_{i=1}^{n}\sum_{j=1}^{b} w^{R_{ji}} = \sqrt{a_1}/n$.

The new coefficient of concordance is based on the sum of squares of deviations of the column totals in Table 2 around their mean value $(1/n)\sum_{i=1}^{n} w_{\cdot i} = (b/n)\sqrt{a_1}$,

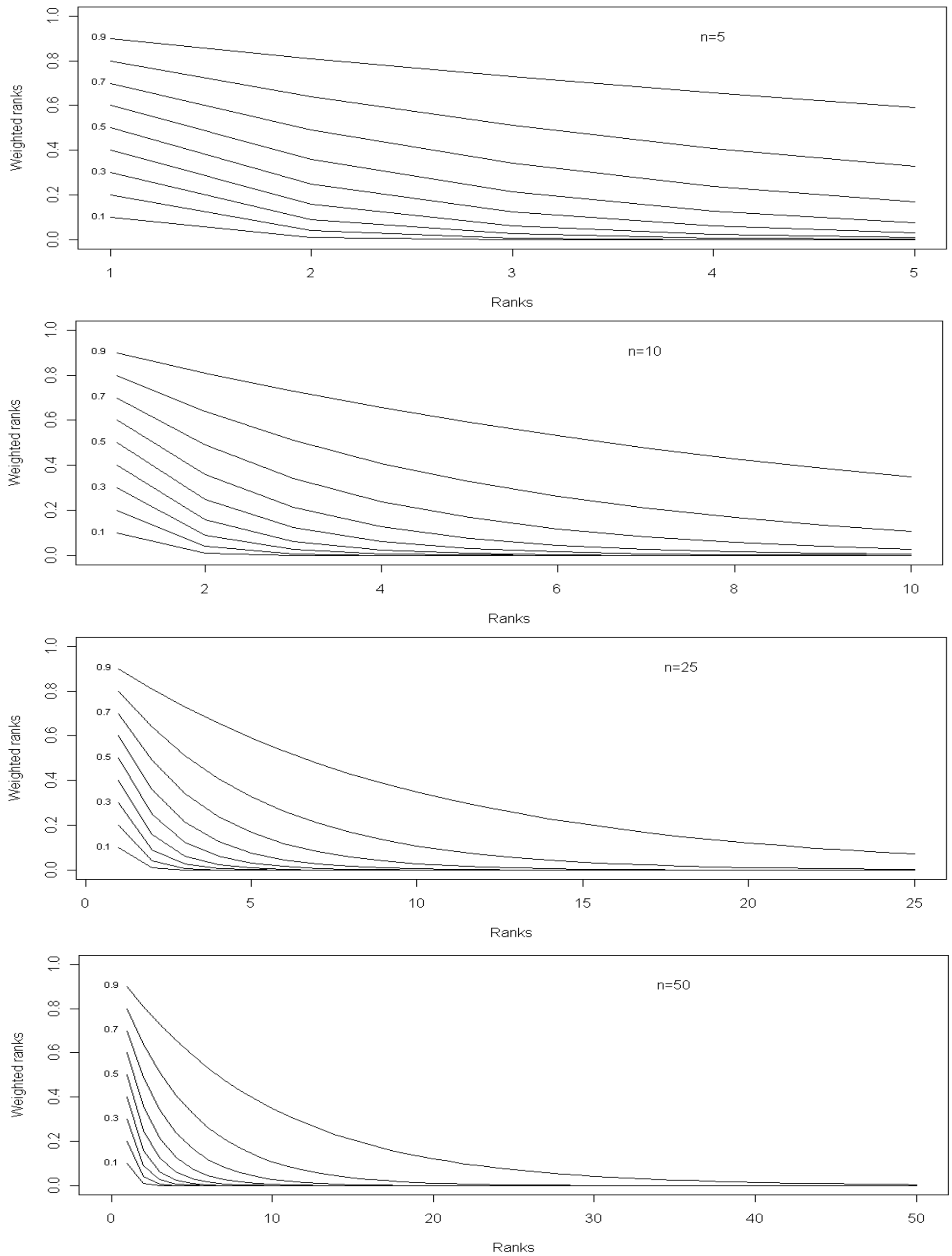

Figure 1. Weighted ranks for $n = 5, 10, 25, 50$ and $w = 0.1(0.1)0.9$

so

$$D = \sum_{i=1}^{n} \left(w_{\cdot i} - \frac{b\sqrt{a_1}}{n} \right)^2 = \sum_{i=1}^{n} w_{\cdot i}^2 - \frac{b^2 a_1}{n} \tag{5}$$

In the ideal case when all observers agree on all the rankings, the sum of the weighted scores assigned to the ith object taken over all b groups of rankings is $w_{\cdot i} = bw^i$, $i = 1, \ldots, n$. Again by taking the sum of squares of deviations of these values $w_{\cdot i}$ around their mean, the maximum value that D can have is

$$\max D = \sum_{i=1}^{n} \left(bw^i - \frac{b\sqrt{a_1}}{n} \right)^2 = b^2 \sum_{i=1}^{n} w^{2i} - \frac{b^2 a_1}{n} = b^2 a_2 - \frac{b^2 a_1}{n} \tag{6}$$

where $a_2 = \sum_{i=1}^{n} w^{2i} = w^2(1 - w^{2n})/(1 - w^2)$. Then from (5) and (6), the new weighted rank concordance coefficient C_w is given by

$$C_w = \frac{D}{\max D} = \frac{(1/b^2)\sum_{i=1}^{n} w_{\cdot i}^2 - (a_1/n)}{a_2 - (a_1/n)} \tag{7}$$

C_w can take values between 0 and 1, where the value 1 is achieved when we have perfect agreement of all rankings, i.e. all observers agree on the ranking of the n subjects. C_w can take the value 0 only if $n = b$ and if each ranking occurs only once in any row or column, so each row or column is a permutation of $1, 2, \ldots, n$, hence the data form a Latin Square. In this situation we have the same number of objects and observers and no object receives the same rank from more than one observer, so each object receives a permutation of $1, 2, \ldots, b = n$. This is logical since there is no agreement between observers on any object. In this case, $w_{\cdot i} = \sqrt{a_1}$ and therefore, from (7), $C_w = 0$ for all $0 < w < 1$.

There is a relationship between C_w and the average value of the weighted rank correlation coefficients R_w^{av}, calculated using (4), between the $\binom{b}{2}$ possible pairs, which is given by

$$R_w^{av} = \frac{bC_w - 1}{b - 1} \tag{8}$$

The proof of this relationship is given in the appendix. One could argue that it is possible to use all pairwise rank coefficients and perform tests of no agreement, instead of the overall concordance measure C_w. However, such tests may lead to increased type I error, as explained in detail by Gibbons & Chakraborti [3, p. 452].

4. The exact and limiting distributions of C_w

In order to use C_w to test the null hypothesis of no agreement between the rankings, one needs to find the distribution of this concordance coefficient under the null hypothesis. In Table 3, the exact quantiles for the weighted rank coefficient of concordance C_w are given for $n = 3$, $b = 3, 4, 5, 6$ and for $n = 4$, $b = 3, 4$. The exact distribution is calculated over all possible permutations $(n!)^b$, for larger values of n and b this becomes computationally difficult. Approximate quantiles for the weighted rank coefficient of concordance C_w, for $n = 4$, $b = 5, 6$ and for $n = 5, 6, 7$, $b = 3, 4, 5, 6$, are summarized in Tables 4 and 5. These approximate quantiles are obtained by simulating 100,000 permutations.

For large values of b and n, the asymptotic distribution of C_w under the null hypothesis is given in the following theorem, the proof is given in the appendix.

Theorem 4.1 *As $b \to \infty$, and under the assumption of random assignment of ranks by all observers, so agreement, the statistic $b(n-1)C_w$ is asymptotically chi-squared distributed with $(n-1)$ degrees of freedom.*

In the case of ties, the average scores can be used with the ordinary F statistic

Table 3. Exact quantiles for the weighted rank coefficient of concordance, C_w.

	$n=3,b=3$					$n=3,b=4$				
w	90%	92.5%	95%	97.5%	99%	90%	92.5%	95%	97.5%	99%
0.1	0.9940	0.9940	0.9940	1	1	0.4713	0.4713	0.4932	0.9932	0.9949
0.2	0.9785	0.9785	0.9785	1	1	0.4980	0.4980	0.5464	0.9758	0.9819
0.3	0.9568	0.9568	0.9568	1	1	0.5184	0.5184	0.5953	0.9514	0.9636
0.4	0.9316	0.9316	0.9316	1	1	0.5337	0.5337	0.6394	0.9231	0.9423
0.5	0.9048	0.9048	0.9048	1	1	0.5446	0.5446	0.6786	0.8929	0.9196
0.6	0.8776	0.8776	0.8776	1	1	0.5523	0.5523	0.7130	0.8622	0.8967
0.7	0.8508	0.8508	0.8508	1	1	0.5574	0.5574	0.7432	0.8322	0.8741
0.8	0.8251	0.8251	0.8251	1	1	0.5605	0.5605	0.7695	0.8033	0.8525
0.9	0.8007	0.8007	0.8007	1	1	0.5620	0.5620	0.7758	0.7924	0.8319
	$n=3,b=5$					$n=3,b=6$				
w	90%	92.5%	95%	97.5%	99%	90%	92.5%	95%	97.5%	99%
0.1	0.4941	0.5200	0.5395	0.5524	0.9935	0.3476	0.3634	0.5503	0.5886	0.6036
0.2	0.4658	0.5200	0.5510	0.5781	0.9768	0.3575	0.3871	0.5188	0.5887	0.6156
0.3	0.4708	0.5200	0.5563	0.5977	0.9534	0.3639	0.4053	0.4970	0.5851	0.6211
0.4	0.4523	0.5262	0.5569	0.6123	0.9262	0.3675	0.4188	0.4872	0.5833	0.6218
0.5	0.4514	0.5200	0.5543	0.6229	0.8971	0.3690	0.4405	0.4643	0.5833	0.6190
0.6	0.4710	0.5127	0.5494	0.6302	0.8678	0.3690	0.4354	0.4643	0.5731	0.6599
0.7	0.4871	0.5047	0.5430	0.6351	0.8389	0.3680	0.4182	0.4844	0.5620	0.6899
0.8	0.4964	0.5003	0.5357	0.6380	0.8111	0.3661	0.3975	0.5014	0.5628	0.6926
0.9	0.4881	0.5111	0.5280	0.6396	0.7848	0.3638	0.3785	0.5157	0.5741	0.6940
	$n=4,b=3$					$n=4,b=4$				
w	90%	92.5%	95%	97.5%	99%	90%	92.5%	95%	97.5%	99%
0.1	0.4557	0.4868	0.9915	0.9943	0.9949	0.4967	0.5141	0.5287	0.5463	0.9921
0.2	0.5004	0.5632	0.9679	0.9784	0.9826	0.4888	0.5222	0.5477	0.5844	0.9708
0.3	0.5397	0.6322	0.9330	0.9544	0.9672	0.4914	0.5264	0.5613	0.6136	0.9394
0.4	0.5723	0.6906	0.8909	0.9247	0.9517	0.4978	0.5339	0.5663	0.6512	0.9017
0.5	0.5981	0.7372	0.8454	0.8918	0.9382	0.5000	0.5348	0.5783	0.6696	0.8565
0.6	0.6172	0.7721	0.7996	0.8577	0.9274	0.5048	0.5348	0.5885	0.6806	0.8145
0.7	0.6406	0.7555	0.7965	0.8359	0.9196	0.5000	0.5281	0.5988	0.6854	0.7908
0.8	0.6743	0.7232	0.7915	0.8257	0.9146	0.5006	0.5409	0.5997	0.6876	0.7741
0.9	0.6775	0.6859	0.7612	0.8199	0.9119	0.5000	0.5397	0.5920	0.6688	0.7540

for the two-way layout, as this statistic automatically corrects for ties as shown by Iman and Davenport [7] for the Friedman test. The F distribution with $n-1$ and $(b-1)(n-1)$ degrees of freedom is used as approximation to the exact distribution of C_w, where

$$F = \frac{(b-1)C_w}{1-C_w} \sim F_{(n-1),(b-1)(n-1)} \tag{9}$$

and C_w can be written as

$$C_w = \frac{F}{(b-1)+F} \tag{10}$$

The proof of (9) is given in the appendix.

5. Simulation study

A simulation study has been carried out to compare the performance of Kendall's, the top-down and the new concordance coefficients in detecting the agreement between the top rankings. Three simulation scenarios are considered: (i) The first scenario is conducted to study type I error of the three concordance coefficients, while the second and the third scenarios are conducted to compare the power of the three concordance coefficients, namely (ii) a non-directional rank agreement

Table 4. Approximate quantiles for the weighted rank coefficient of concordance, C_w.

	$n=4, b=5$					$n=4, b=6$				
w	90%	92.5%	95%	97.5%	99%	90%	92.5%	95%	97.5%	99%
0.1	0.3690	0.3917	0.5509	0.5764	0.6039	0.3397	0.3596	0.3939	0.4275	0.6168
0.2	0.3748	0.4253	0.5267	0.5813	0.6242	0.3400	0.3624	0.3938	0.4548	0.5992
0.3	0.3825	0.4481	0.5041	0.5773	0.6345	0.3344	0.3616	0.3981	0.4714	0.5802
0.4	0.3991	0.4417	0.4841	0.5753	0.6358	0.3319	0.3587	0.4014	0.4847	0.5615
0.5	0.3990	0.4379	0.4797	0.5715	0.6383	0.3333	0.3604	0.4068	0.4802	0.5614
0.6	0.3993	0.4337	0.4827	0.5682	0.6475	0.3342	0.3661	0.4129	0.4733	0.5659
0.7	0.4009	0.4340	0.4840	0.5611	0.6494	0.3375	0.3709	0.4112	0.4740	0.5623
0.8	0.4044	0.4328	0.4896	0.5583	0.6482	0.3402	0.3728	0.4132	0.4807	0.5603
0.9	0.4085	0.4365	0.4907	0.5637	0.6571	0.3424	0.3714	0.4100	0.4785	0.5588
	$n=5, b=3$					$n=5, b=4$				
w	90%	92.5%	95%	97.5%	99%	90%	92.5%	95%	97.5%	99%
0.1	0.4759	0.4966	0.4997	0.9941	0.9947	0.5105	0.5198	0.5354	0.5648	0.5828
0.2	0.5121	0.5399	0.5531	0.9709	0.9798	0.4907	0.5029	0.5313	0.5862	0.6256
0.3	0.5479	0.5717	0.6034	0.9428	0.9566	0.4686	0.4940	0.5373	0.5990	0.6664
0.4	0.5750	0.5971	0.6454	0.9006	0.9270	0.4562	0.4853	0.5460	0.6069	0.6983
0.5	0.5833	0.6177	0.6834	0.8536	0.8925	0.4540	0.4850	0.5422	0.6069	0.7169
0.6	0.5904	0.6390	0.7246	0.8143	0.8558	0.4550	0.4939	0.5424	0.6076	0.7264
0.7	0.5975	0.6567	0.7051	0.7869	0.8430	0.4626	0.4984	0.5429	0.6151	0.7136
0.8	0.6117	0.6492	0.6953	0.7757	0.8362	0.4646	0.4966	0.5401	0.6146	0.6898
0.9	0.6073	0.6451	0.6868	0.7552	0.8324	0.4652	0.4978	0.5446	0.6078	0.6835
	$n=5, b=5$					$n=5, b=6$				
w	90%	92.5%	95%	97.5%	99%	90%	92.5%	95%	97.5%	99%
0.1	0.3434	0.3868	0.4144	0.5833	0.6066	0.2810	0.3617	0.3819	0.4183	0.4704
0.2	0.3534	0.3804	0.4217	0.5577	0.6053	0.3122	0.3460	0.3815	0.4219	0.4889
0.3	0.3565	0.3859	0.4309	0.5315	0.6130	0.3108	0.3368	0.3739	0.4253	0.5042
0.4	0.3644	0.3966	0.4484	0.5172	0.6100	0.3092	0.3356	0.3697	0.4239	0.5144
0.5	0.3699	0.4016	0.4430	0.5043	0.5968	0.3119	0.3369	0.3701	0.4276	0.5042
0.6	0.3727	0.4022	0.4401	0.5032	0.5880	0.3124	0.3382	0.3725	0.4313	0.5024
0.7	0.3733	0.4017	0.4392	0.5037	0.5756	0.3134	0.3385	0.3721	0.4287	0.4913
0.8	0.3761	0.4037	0.4419	0.5054	0.5766	0.3149	0.3392	0.3739	0.4274	0.4932
0.9	0.3767	0.4059	0.4427	0.5021	0.5780	0.3155	0.3408	0.3748	0.4270	0.4903

scenario and (iii) a directional rank agreement scenario. We followed Legendre [10] and Teles [18] in generating these simulation scenarios. The simulation study results are based on 10,000 replications, where for all scenarios we consider $n = 10, 20, 30, 50, 100$ and $b = 3, 4, 5, 6$. Thereafter, the b sets of simulated observations are converted into ranks. In more detail:

(i) in order to estimate type I error, b independent random samples of size n are generated from the standard normal distribution. Type I error is estimated as the percentage of rejections of the null hypothesis (of no agreement) when the data are in agreement with this hypothesis. For a statistical test to be valid, its type I error should not exceed the nominal significance level α. We compare the type I errors of Kendall's, the top-down and the new concordance coefficients, for ease of presentation we selected $w = 0.4, 0.6, 0.7, 0.9$. Figure 2 shows that type I errors of all tests do not exceed the nominal significant level $\alpha = 0.05$. While for small n all tests have similar type I errors, they differ as n increases. Overall Kendall's concordance coefficient has the lowest type I errors, however, the proposed weighted concordance coefficient ($C_{0.9}$) has small type I errors for small samples. For larger b, the differences between type I errors become even smaller. We also notice overall that $C_{0.9}$ has lower type I errors than $C_{0.4}$ which is logical as the values of the data decrease with smaller weight, see Figure 1.

(ii) The non-directional rank agreement scenario is conducted as follows: For the first set of observations, a standard normal distribution sample is generated. For the remaining $b-1$ sets of observations, normal distributions samples with mean zero and standard deviations $\sigma = 0.5, 1, 2, 3$ are simulated and then added to the

Table 5. Approximate quantiles for the weighted rank coefficient of concordance, C_w.

	$n=6, b=3$					$n=6, b=4$				
w	90%	92.5%	95%	97.5%	99%	90%	92.5%	95%	97.5%	99%
0.1	0.4972	0.4977	0.5222	0.9919	0.9945	0.4638	0.5300	0.5422	0.5714	0.5884
0.2	0.5219	0.5264	0.5689	0.9677	0.9776	0.4740	0.5083	0.5263	0.5680	0.6176
0.3	0.5380	0.5518	0.6025	0.9282	0.9489	0.4615	0.4887	0.5111	0.5800	0.6358
0.4	0.5476	0.5741	0.6278	0.8755	0.9116	0.4426	0.4704	0.5062	0.5840	0.6473
0.5	0.5546	0.5925	0.6431	0.8139	0.8670	0.4344	0.4632	0.5034	0.5814	0.6612
0.6	0.5634	0.5999	0.6548	0.7656	0.8298	0.4341	0.4639	0.5060	0.5735	0.6526
0.7	0.5694	0.6057	0.6611	0.7326	0.8052	0.4355	0.4660	0.5076	0.5722	0.6518
0.8	0.5763	0.6102	0.6550	0.7141	0.7862	0.4379	0.4684	0.5074	0.5682	0.6348
0.9	0.5745	0.6066	0.6496	0.7124	0.7831	0.4404	0.4691	0.5069	0.5634	0.6285
	$n=6, b=5$					$n=6, b=6$				
w	90%	92.5%	95%	97.5%	99%	90%	92.5%	95%	97.5%	99%
0.1	0.3379	0.3490	0.4009	0.4467	0.6058	0.2752	0.2876	0.3821	0.4106	0.4552
0.2	0.3446	0.3643	0.3959	0.4597	0.5877	0.2825	0.3062	0.3638	0.4123	0.4575
0.3	0.3464	0.3673	0.4018	0.4817	0.5735	0.2901	0.3175	0.3525	0.4063	0.4625
0.4	0.3470	0.3705	0.4075	0.4848	0.5542	0.2941	0.3178	0.3489	0.4012	0.4661
0.5	0.3494	0.3741	0.4131	0.4734	0.5410	0.2953	0.3175	0.3483	0.4000	0.4667
0.6	0.3521	0.3781	0.4133	0.4697	0.5417	0.2957	0.3179	0.3478	0.3967	0.4583
0.7	0.3549	0.3794	0.4132	0.4677	0.5347	0.2974	0.3190	0.3486	0.3970	0.4541
0.8	0.3564	0.3817	0.4139	0.4665	0.5294	0.2993	0.3215	0.3520	0.3992	0.4539
0.9	0.3558	0.3809	0.4133	0.4661	0.5281	0.2990	0.3211	0.3508	0.3966	0.4526
	$n=7, b=3$					$n=7, b=4$				
w	90%	92.5%	95%	97.5%	99%	90%	92.5%	95%	97.5%	99%
0.1	0.4914	0.5127	0.5154	0.5417	0.9925	0.4008	0.4534	0.5460	0.5595	0.5892
0.2	0.5023	0.5366	0.5463	0.5914	0.9715	0.4159	0.4968	0.5278	0.5541	0.6104
0.3	0.5187	0.5542	0.5760	0.6498	0.9401	0.4112	0.4774	0.5090	0.5472	0.6221
0.4	0.5349	0.5665	0.5989	0.6886	0.9025	0.4233	0.4546	0.4896	0.5427	0.6289
0.5	0.5400	0.5731	0.6185	0.7167	0.8486	0.4192	0.4455	0.4811	0.5422	0.6257
0.6	0.5426	0.5764	0.6243	0.7250	0.8023	0.4193	0.4454	0.4810	0.5436	0.6171
0.7	0.5479	0.5814	0.6287	0.6973	0.7615	0.4203	0.4477	0.4844	0.5428	0.6114
0.8	0.5521	0.5825	0.6202	0.6802	0.7456	0.4207	0.4469	0.4816	0.5361	0.5991
0.9	0.5482	0.5785	0.6163	0.6735	0.7374	0.4229	0.4486	0.4827	0.5350	0.5944
	$n=7, b=5$					$n=7, b=6$				
w	90%	92.5%	95%	97.5%	99%	90%	92.5%	95%	97.5%	99%
0.1	0.3420	0.3528	0.3691	0.4375	0.6107	0.2810	0.2907	0.3061	0.4075	0.4343
0.2	0.3359	0.3584	0.3847	0.4316	0.5863	0.2775	0.2943	0.3221	0.3956	0.4416
0.3	0.3375	0.3597	0.3859	0.4406	0.5621	0.2780	0.2975	0.3332	0.3853	0.4403
0.4	0.3360	0.3585	0.3885	0.4484	0.5334	0.2817	0.3032	0.3332	0.3795	0.4374
0.5	0.3369	0.3599	0.3908	0.4508	0.5151	0.2826	0.3026	0.3293	0.3757	0.4314
0.6	0.3376	0.3603	0.3916	0.4445	0.5049	0.2835	0.3035	0.3301	0.3738	0.4287
0.7	0.3394	0.3619	0.3928	0.4419	0.5051	0.2848	0.3045	0.3309	0.3739	0.4279
0.8	0.3412	0.3632	0.3930	0.4421	0.4980	0.2853	0.3043	0.3296	0.3710	0.4221
0.9	0.3430	0.3651	0.3943	0.4387	0.4940	0.2870	0.3059	0.3302	0.3718	0.4214

first set of observations to create some agreement them, where larger values of σ correspond to lower degrees of agreement.

(iii) The directional rank agreement scenario is carried out as follows: For the first set of observations, a standard normal distribution sample is generated. This sample is sorted ascendingly and divided into two halves, the first half corresponding to the lower ranks and the second to the higher ranks. For the first half of the remaining $b-1$ sets of observations, normal distributions samples (of size $n/2$) with mean zero and standard deviations $\sigma = 0.5, 1, 2, 3$ are simulated and then added to the first (sorted) set of observations to create some agreement. The second half of the remaining $b-1$ sets of observations consist of standard normally distributed samples

The performance (power) of the three concordance coefficients are assessed, at significance level $\alpha = 0.05$, by the percentage of rejections of the null hypothesis when the null hypothesis is false. Based on 10,000 replications, the simula-

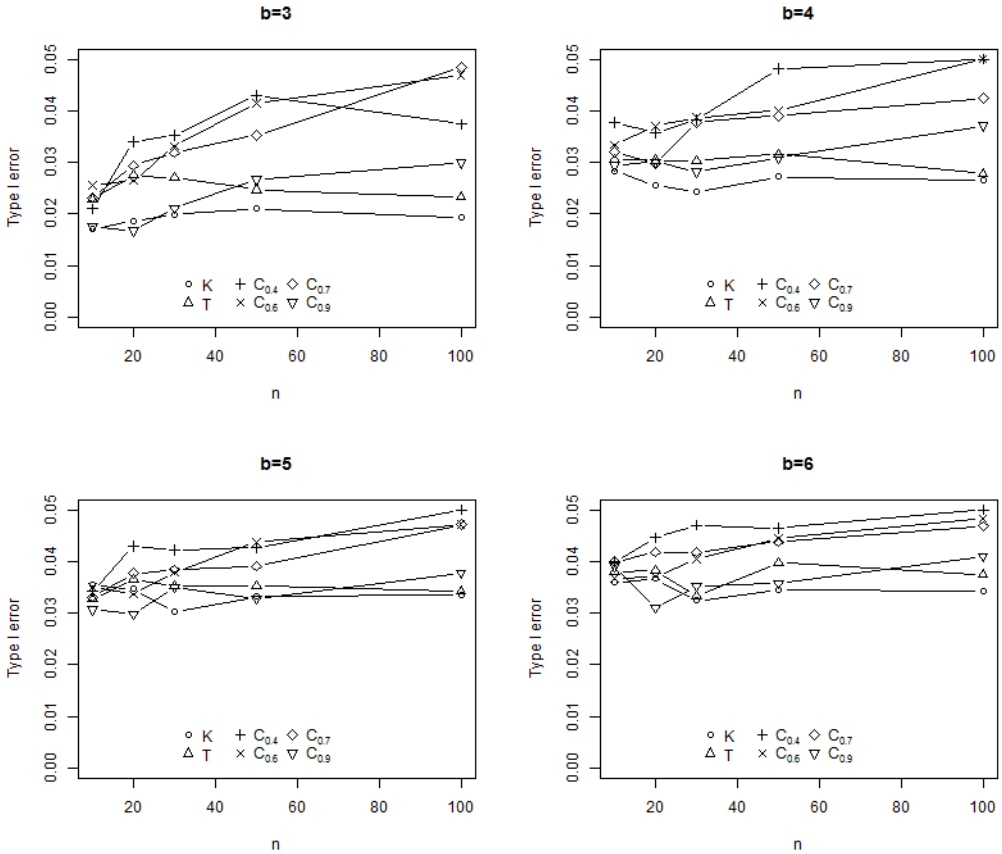

Figure 2. Type I error comparison

tion results for the non-directional rank agreement scenario are summarised in Tables 6-9 and for the directional rank agreement scenario in Tables 10-13. As expected, Kendall's concordance coefficient performs better in the non-directional agreement scenario while the top-down and the new weighted concordance coefficients achieved better power in the directional agreement scenario, similar results were obtained by Teles [18] comparing Kendall's and the top-down concordance coefficients. The performance of the $C_{0.9}$, under the non-directional agreement scenario, is very close to Kendall's and the top-down concordance coefficients except when σ is large. For the directional agreement scenario, the performance of the new weighted concordance coefficient exceeds Kendall's concordance coefficient especially for large values of w, while it performs better or similar to the top-down concordance coefficient.

6. Examples

In this section, three examples are provided to illustrate the new proposed weighted rank concordance coefficient, where the third example considers real data on financial markets indices.

Example 6.1 Let us consider the scenario when three financial experts are asked to rank four investment projects in terms of their profitability. Their rankings are summarized in Table 14. Case A in this table shows that all experts agree that project 1 is the best investment project while the experts' opinions vary with regard to the other projects. In order to illustrate the important feature of the

Table 6. Power comparisons of K, T and C_w for the non-directional scenario, $b = 3$.

σ	n	K	T	$C_{0.1}$	$C_{0.2}$	$C_{0.3}$	$C_{0.4}$	$C_{0.5}$	$C_{0.6}$	$C_{0.7}$	$C_{0.8}$	$C_{0.9}$
0.5	10	0.9857	0.9638	0.4448	0.6171	0.7263	0.8017	0.8671	0.9207	0.9523	0.9750	0.9853
	20	1	0.9995	0.8211	0.8215	0.8462	0.8969	0.9441	0.9782	0.9945	0.9992	1
	30	1	1	0.8326	0.8250	0.8741	0.9253	0.9644	0.9865	0.9983	0.9999	1
	50	1	1	0.7855	0.8226	0.8874	0.9378	0.9761	0.9938	0.9997	1	1
	100	1	1	0.7447	0.8374	0.9007	0.9560	0.9860	0.9964	0.9998	1	1
1	10	0.7791	0.6967	0.2133	0.2901	0.3880	0.4659	0.5324	0.6021	0.6683	0.7258	0.7704
	20	0.9818	0.9483	0.5225	0.5210	0.5289	0.5932	0.6597	0.7482	0.8506	0.9285	0.9722
	30	0.9990	0.9945	0.5715	0.5784	0.6049	0.6557	0.7232	0.8017	0.8935	0.9664	0.9951
	50	1	1	0.5232	0.5252	0.5885	0.6603	0.7491	0.8542	0.9308	0.9853	0.9993
	100	1	1	0.4415	0.5044	0.5622	0.6799	0.7797	0.8732	0.9512	0.9925	0.9998
2	10	0.3004	0.2738	0.0735	0.0977	0.1401	0.1844	0.2175	0.2514	0.2679	0.2963	0.3154
	20	0.6086	0.5255	0.2299	0.2226	0.2286	0.2425	0.2778	0.3279	0.4008	0.4960	0.5799
	30	0.8092	0.7034	0.3173	0.3284	0.3384	0.3333	0.3280	0.3705	0.4486	0.5731	0.7321
	50	0.9610	0.9034	0.2578	0.2721	0.2942	0.3150	0.3606	0.4276	0.5110	0.6576	0.8563
	100	0.9995	0.9979	0.1947	0.2069	0.2278	0.2888	0.3612	0.4385	0.5489	0.7022	0.9262
3	10	0.1546	0.1508	0.0441	0.0506	0.0775	0.1050	0.1237	0.1363	0.1493	0.1588	0.1524
	20	0.3168	0.2828	0.1412	0.1329	0.1407	0.1468	0.1607	0.1805	0.2139	0.2640	0.3049
	30	0.4836	0.4037	0.2341	0.2373	0.2430	0.2083	0.1917	0.2034	0.2443	0.3157	0.4117
	50	0.7058	0.6046	0.1772	0.1795	0.1898	0.2120	0.2319	0.2467	0.2942	0.3697	0.5413
	100	0.9535	0.8889	0.1140	0.1236	0.1299	0.1599	0.2062	0.2589	0.3084	0.4092	0.6334

Table 7. Power comparisons of K, T and C_w for the non-directional scenario, $b = 4$.

σ	n	K	T	$C_{0.1}$	$C_{0.2}$	$C_{0.3}$	$C_{0.4}$	$C_{0.5}$	$C_{0.6}$	$C_{0.7}$	$C_{0.8}$	$C_{0.9}$
0.5	10	0.9978	0.9925	0.7712	0.7922	0.8614	0.9202	0.9529	0.9814	0.9911	0.9956	0.9980
	20	1	1	0.7654	0.8905	0.9369	0.9662	0.9852	0.9958	0.9991	1	1
	30	1	1	0.9011	0.9235	0.9521	0.9786	0.9927	0.9979	0.9999	1	1
	50	1	1	0.9196	0.9307	0.9651	0.9827	0.9970	0.9997	1	1	1
	100	1	1	0.8859	0.9294	0.9704	0.9892	0.9981	0.9998	1	1	1
1	10	0.9126	0.8544	0.4720	0.4988	0.5564	0.6415	0.7118	0.7784	0.8371	0.8814	0.9116
	20	0.9975	0.9905	0.4054	0.5673	0.6625	0.7367	0.8106	0.8913	0.9475	0.9831	0.9954
	30	1	0.9997	0.5759	0.6614	0.6995	0.7778	0.8511	0.9208	0.9661	0.9938	1
	50	1	1	0.7020	0.7029	0.7408	0.8120	0.8788	0.9382	0.9841	0.9986	1
	100	1	1	0.6079	0.6456	0.7296	0.8123	0.8948	0.9546	0.9885	0.9992	1
2	10	0.4837	0.4152	0.2190	0.2295	0.2508	0.2883	0.3229	0.3621	0.3979	0.4496	0.4791
	20	0.7893	0.7047	0.1426	0.2161	0.2692	0.3218	0.3767	0.4653	0.5641	0.6599	0.7386
	30	0.9256	0.8597	0.2329	0.2867	0.3182	0.3518	0.4193	0.503	0.6141	0.7499	0.8816
	50	0.9920	0.9760	0.3964	0.3940	0.4144	0.4067	0.4548	0.5345	0.6513	0.8065	0.9513
	100	1	0.9998	0.3048	0.3078	0.3430	0.3887	0.4694	0.5685	0.6866	0.8455	0.9774
3	10	0.2581	0.2294	0.1259	0.1441	0.1551	0.1696	0.1849	0.1978	0.2149	0.2408	0.2579
	20	0.4698	0.4062	0.0779	0.1247	0.1513	0.1800	0.2163	0.2499	0.2942	0.3754	0.4440
	30	0.6324	0.5550	0.1256	0.1615	0.1820	0.2037	0.2247	0.2827	0.3406	0.4444	0.5599
	50	0.8485	0.7629	0.2787	0.2810	0.2881	0.2394	0.2643	0.3064	0.3729	0.4813	0.7071
	100	0.9893	0.9615	0.1855	0.1884	0.2122	0.2326	0.2713	0.3213	0.3993	0.5283	0.7787

new weighted rank concordance coefficient that the emphasis is on top rankings, we also consider Case B in Table 14 where all experts agree that project 4 is the worst project to invest in, but their rankings vary over the first three projects.

For both cases, Kendall's coefficient of concordance is $K = 0.6$ and the p-value is 0.175. So we do not reject the null hypothesis of no agreement between the rankings for both cases at significance level 10%. However, the top-down coefficient of concordance is $T = 0.816$ for Case A with p-value $= 0.062$ and $T = 0.391$ for Case B with p-value $= 0.373$. So for Case A, we reject the null hypothesis at significance level 10%, while for Case B we do not reject the null hypothesis of no agreement between the rankings at significance level 10%. In Table 15, the weighted rank coefficient of concordance C_w is presented for several values of $w = 0.1(0.1)0.9$ along with the corresponding p-values for both cases. So for Case B, we do not

Table 8. Power comparisons of K, T and C_w for the non-directional scenario, $b = 5$.

σ	n	K	T	$C_{0.1}$	$C_{0.2}$	$C_{0.3}$	$C_{0.4}$	$C_{0.5}$	$C_{0.6}$	$C_{0.7}$	$C_{0.8}$	$C_{0.9}$
0.5	10	0.9994	0.9988	0.8481	0.9041	0.9383	0.9664	0.9837	0.9935	0.9974	0.9991	0.9995
	20	1	1	0.8898	0.9398	0.9721	0.9887	0.9959	0.9997	1	1	1
	30	1	1	0.9051	0.9615	0.9832	0.9923	0.9988	0.9999	1	1	1
	50	1	1	0.9644	0.9742	0.9883	0.9958	0.9993	0.9999	1	1	1
	100	1	1	0.9523	0.9757	0.9908	0.9974	0.9997	1	1	1	1
1	10	0.9603	0.9280	0.5875	0.6279	0.6764	0.7487	0.8148	0.8800	0.9154	0.9401	0.9553
	20	0.9992	0.9974	0.6120	0.6665	0.7563	0.8310	0.8989	0.9497	0.9818	0.9955	0.9994
	30	1	1	0.5685	0.7142	0.7917	0.8644	0.9212	0.9657	0.9932	0.9994	0.9999
	50	1	1	0.7449	0.7845	0.8234	0.8849	0.9424	0.9779	0.9959	0.9996	1
	100	1	1	0.7605	0.7650	0.8340	0.9033	0.9529	0.9855	0.9975	0.9998	1
2	10	0.5954	0.5252	0.2657	0.2849	0.2986	0.3508	0.4043	0.4511	0.5108	0.5627	0.5889
	20	0.8803	0.8019	0.2800	0.2887	0.3260	0.4098	0.4839	0.5651	0.6653	0.7676	0.8496
	30	0.9655	0.9346	0.2028	0.2931	0.3683	0.4282	0.5141	0.5884	0.7159	0.8478	0.9388
	50	0.9982	0.9917	0.3278	0.3811	0.4026	0.4515	0.5365	0.6348	0.7551	0.8888	0.9822
	100	1	1	0.4123	0.4095	0.4302	0.4998	0.5589	0.6583	0.7853	0.9204	0.9958
3	10	0.3388	0.2875	0.1648	0.1746	0.1780	0.1897	0.2245	0.2563	0.2828	0.3042	0.3322
	20	0.5875	0.5079	0.1751	0.1567	0.1859	0.2248	0.2702	0.3321	0.3858	0.4603	0.5444
	30	0.7504	0.6723	0.1099	0.1517	0.2007	0.2487	0.2871	0.3343	0.4176	0.5353	0.6816
	50	0.9218	0.8573	0.1771	0.2248	0.2293	0.2554	0.3034	0.3526	0.4479	0.5849	0.7976
	100	0.9969	0.9866	0.2661	0.2657	0.2753	0.3161	0.3167	0.3722	0.4645	0.6315	0.8712

Table 9. Power comparisons of K, T and C_w for the non-directional scenario, $b = 6$.

σ	n	K	T	$C_{0.1}$	$C_{0.2}$	$C_{0.3}$	$C_{0.4}$	$C_{0.5}$	$C_{0.6}$	$C_{0.7}$	$C_{0.8}$	$C_{0.9}$
0.5	10	0.9998	0.9994	0.9023	0.9393	0.9651	0.9853	0.9939	0.9971	0.9991	0.9996	0.9998
	20	1	1	0.9506	0.9679	0.9873	0.9960	0.9985	1	1	1	1
	30	1	1	0.9541	0.9812	0.9927	0.9982	0.9994	1	1	1	1
	50	1	1	0.9738	0.9887	0.9957	0.9984	0.9998	1	1	1	1
	100	1	1	0.9829	0.9906	0.9974	0.9996	0.9999	1	1	1	1
1	10	0.9787	0.9577	0.6241	0.6837	0.7535	0.8338	0.8892	0.9231	0.9531	0.9707	0.9766
	20	0.9999	0.9996	0.6944	0.7698	0.8341	0.8932	0.9459	0.9754	0.9917	0.9992	0.9998
	30	1	1	0.7179	0.7824	0.8655	0.9152	0.9578	0.9876	0.9979	0.9998	1
	50	1	1	0.7274	0.8325	0.8844	0.9289	0.9722	0.9920	0.9994	0.9999	1
	100	1	1	0.8417	0.8493	0.8960	0.9429	0.9761	0.9938	0.9994	1	1
2	10	0.6772	0.6203	0.2803	0.3087	0.3550	0.4052	0.4709	0.5301	0.5876	0.6378	0.6753
	20	0.9274	0.8787	0.3234	0.3784	0.4193	0.4772	0.5610	0.6529	0.7433	0.8434	0.9086
	30	0.9853	0.9644	0.3410	0.3560	0.4259	0.5025	0.5881	0.6908	0.7955	0.9039	0.9708
	50	0.9998	0.9974	0.2616	0.3664	0.4500	0.5262	0.6149	0.7193	0.8342	0.9407	0.9942
	100	1	1	0.5129	0.5087	0.5004	0.5478	0.6224	0.7364	0.8503	0.9585	0.9987
3	10	0.3967	0.3495	0.1614	0.1752	0.1913	0.2386	0.2601	0.2932	0.3388	0.3702	0.3963
	20	0.6675	0.5828	0.1912	0.2171	0.2313	0.2652	0.3064	0.3802	0.4450	0.5399	0.6301
	30	0.8238	0.7407	0.2095	0.1966	0.2309	0.2781	0.3231	0.3947	0.4842	0.6207	0.7556
	50	0.9572	0.9117	0.1291	0.1876	0.2426	0.2860	0.3368	0.4085	0.5186	0.6711	0.8651
	100	0.9990	0.9953	0.3453	0.3486	0.2935	0.3111	0.3564	0.4341	0.5327	0.7043	0.9254

reject the null hypothesis of no agreement between the rankings for any value of w while for Case A we reject the null hypothesis at significance level 10% for $w \leq 0.8$.

This example shows that Kendall's coefficient of concordance K does not reflect the agreement on the top ranking as it gave the same answer to both cases. However, the top-down coefficient of concordance T and the new weighted rank coefficient of concordance C_w are more sensitive to the agreement on the top rankings. The new weighted rank coefficient of concordance C_w gives the option of choosing the weights w to reflect the desired level of focus on the top rankings.

Example 6.2 Consider a computer model with seven input variables ranked by each of six different measures, as shown in Table 16. This data set was used by Iman and Conover [6] to introduce the top-down concordance coefficient T.

Table 10. Power comparisons of K, T and C_w for the directional scenario, $b = 3$.

σ	n	K	T	$C_{0.1}$	$C_{0.2}$	$C_{0.3}$	$C_{0.4}$	$C_{0.5}$	$C_{0.6}$	$C_{0.7}$	$C_{0.8}$	$C_{0.9}$
0.5	10	0.4757	0.5668	0.2886	0.3621	0.4335	0.4968	0.5284	0.5621	0.5854	0.5711	0.5328
	20	0.8125	0.8574	0.6215	0.6195	0.6448	0.6844	0.7390	0.7881	0.8318	0.8641	0.8653
	30	0.9395	0.9569	0.6878	0.6919	0.7140	0.7682	0.8115	0.8693	0.9079	0.9490	0.9650
	50	0.9967	0.9964	0.6718	0.6975	0.7618	0.8191	0.8721	0.9222	0.9617	0.9879	0.9965
	100	1	1	0.6476	0.7326	0.7895	0.8697	0.9230	0.9674	0.9865	0.9981	1
1	10	0.3155	0.4356	0.1822	0.2248	0.3004	0.3475	0.3945	0.4247	0.4408	0.4098	0.3711
	20	0.6038	0.7389	0.4521	0.4617	0.4820	0.5049	0.5633	0.616	0.6773	0.7288	0.7180
	30	0.7970	0.8857	0.5491	0.5564	0.5736	0.6024	0.6628	0.7063	0.7952	0.8619	0.8884
	50	0.9507	0.9833	0.5048	0.5115	0.5654	0.6302	0.7169	0.7989	0.8734	0.9377	0.9795
	100	0.9991	1	0.4367	0.4832	0.5377	0.6566	0.7547	0.8490	0.9255	0.9806	0.9989
2	10	0.1489	0.2824	0.0979	0.1239	0.1898	0.2364	0.2713	0.2883	0.2946	0.2595	0.2026
	20	0.2988	0.5303	0.2903	0.2884	0.3021	0.3283	0.3894	0.4422	0.5194	0.5628	0.4870
	30	0.4306	0.7135	0.3729	0.3778	0.3892	0.4131	0.4267	0.5057	0.6131	0.7288	0.7255
	50	0.6392	0.9015	0.2936	0.2953	0.3330	0.3746	0.4419	0.5381	0.6668	0.8276	0.9272
	100	0.9032	0.9957	0.2049	0.2237	0.2536	0.3186	0.4122	0.5268	0.6705	0.8486	0.9891
3	10	0.0961	0.2178	0.0751	0.0974	0.1533	0.1930	0.2206	0.2505	0.2329	0.1986	0.1448
	20	0.1790	0.4125	0.2331	0.2305	0.2384	0.2747	0.3076	0.3682	0.4303	0.4678	0.3762
	30	0.2567	0.5683	0.3055	0.3014	0.3131	0.3230	0.3405	0.405	0.5265	0.6412	0.6129
	50	0.3859	0.7962	0.2280	0.2281	0.2538	0.2704	0.3355	0.4155	0.5350	0.7421	0.8776
	100	0.6418	0.9783	0.1446	0.1557	0.1618	0.2151	0.2858	0.3706	0.4912	0.7146	0.9706

Table 11. Power comparisons of K, T and C_w for the directional scenario, $b = 4$.

σ	n	K	T	$C_{0.1}$	$C_{0.2}$	$C_{0.3}$	$C_{0.4}$	$C_{0.5}$	$C_{0.6}$	$C_{0.7}$	$C_{0.8}$	$C_{0.9}$
0.5	10	0.6999	0.7264	0.5310	0.5541	0.5890	0.6280	0.6801	0.7002	0.7297	0.7472	0.7267
	20	0.9364	0.9364	0.5393	0.6793	0.7343	0.7842	0.8341	0.8740	0.9095	0.9404	0.9520
	30	0.9895	0.9847	0.7217	0.7599	0.7964	0.8468	0.8964	0.9285	0.9632	0.9823	0.9909
	50	0.9996	0.9994	0.8165	0.8244	0.8589	0.9019	0.9426	0.9656	0.9852	0.9968	0.9994
	100	1	1	0.7916	0.8329	0.8992	0.9394	0.9696	0.9843	0.9972	0.9995	1
1	10	0.4920	0.5875	0.4042	0.4192	0.4582	0.5083	0.5352	0.5659	0.5911	0.5796	0.5626
	20	0.7913	0.8642	0.3539	0.4902	0.5641	0.6373	0.7002	0.7607	0.8195	0.8538	0.8549
	30	0.9193	0.9600	0.5268	0.6113	0.6477	0.7036	0.7700	0.8324	0.8929	0.9433	0.9576
	50	0.9899	0.9956	0.6834	0.6875	0.7255	0.7770	0.8316	0.8995	0.9493	0.9815	0.9956
	100	0.9999	1	0.5985	0.6320	0.7168	0.7956	0.8733	0.9365	0.9770	0.9971	1
2	10	0.2512	0.3998	0.2746	0.3038	0.3306	0.3735	0.3965	0.4136	0.4210	0.3920	0.3382
	20	0.4412	0.6868	0.1953	0.3065	0.3893	0.4583	0.5414	0.6233	0.6944	0.7288	0.6383
	30	0.5935	0.8495	0.2981	0.3772	0.4244	0.4912	0.5682	0.6893	0.8056	0.8669	0.8637
	50	0.7938	0.9704	0.4584	0.4570	0.4780	0.5157	0.5828	0.7038	0.8329	0.9489	0.9818
	100	0.9707	0.9996	0.3345	0.3415	0.3901	0.4554	0.5506	0.6782	0.8304	0.9614	0.9997
3	10	0.1612	0.3140	0.2356	0.2523	0.2770	0.3166	0.3365	0.3513	0.3403	0.2969	0.2361
	20	0.2699	0.5648	0.1435	0.2267	0.2976	0.3696	0.4639	0.5589	0.6279	0.6358	0.5151
	30	0.3719	0.7375	0.2130	0.2857	0.3203	0.3808	0.4653	0.5915	0.7305	0.8259	0.7675
	50	0.5211	0.9114	0.3640	0.3728	0.3708	0.3856	0.4486	0.5678	0.7517	0.9164	0.9670
	100	0.7840	0.9971	0.2411	0.2491	0.2767	0.3116	0.3954	0.5033	0.6691	0.8965	0.9985

The top-down concordance coefficient for these data is $T = 0.672$ and $b(n-1)T = 24.18$, comparing this with the chi-squared distribution with 6 degrees of freedom, $\chi^2_{6,.95} = 12.59$ (or p-value $= 0.0005$) indicates a strong agreement among the six measures. Kendall's coefficient of concordance is $K = 0.375$ and $b(n-1)K = 13.5$ with p-value $= 0.0357$, this shows moderate agreement among the six measures. One can also use the F approximation,

$$F_T = \frac{(b-1)T}{1-T} = 10.24 \quad \text{and} \quad F_K = \frac{(b-1)K}{1-K} = 3$$

and compare this with F distribution with 6 and 30 degrees of freedom, the p-value is less than 10^{-6} for F_T and equal to 0.0204 for F_K.

Before calculating the new weighted rank coefficient of concordance C_w, let us

Table 12. Power comparisons of K, T and C_w for the directional scenario, $b = 5$.

σ	n	K	T	$C_{0.1}$	$C_{0.2}$	$C_{0.3}$	$C_{0.4}$	$C_{0.5}$	$C_{0.6}$	$C_{0.7}$	$C_{0.8}$	$C_{0.9}$
0.5	10	0.8095	0.8022	0.6126	0.6505	0.6788	0.7198	0.7544	0.784	0.8098	0.8185	0.8296
	20	0.9760	0.9644	0.6847	0.7335	0.7934	0.8408	0.8877	0.9196	0.9464	0.9684	0.9773
	30	0.9964	0.9942	0.6981	0.7995	0.8481	0.8937	0.9294	0.9583	0.9807	0.9898	0.9961
	50	1	0.9997	0.8524	0.8704	0.9028	0.9354	0.9652	0.9829	0.9904	0.9987	0.9995
	100	1	1	0.8829	0.8998	0.9342	0.9668	0.9857	0.9948	0.9990	0.9999	1
1	10	0.6289	0.6800	0.4866	0.5111	0.5394	0.5872	0.6314	0.6685	0.6792	0.6873	0.6690
	20	0.8809	0.9161	0.5479	0.5743	0.6657	0.7229	0.7836	0.8395	0.8953	0.9139	0.9162
	30	0.9664	0.9834	0.5090	0.6436	0.7267	0.7978	0.8508	0.9018	0.9443	0.9695	0.9779
	50	0.9976	0.9994	0.7072	0.7518	0.7911	0.8484	0.9009	0.9453	0.9805	0.9930	0.9989
	100	1	1	0.7369	0.7550	0.8125	0.8808	0.9329	0.9716	0.9939	0.9993	1
2	10	0.3414	0.4929	0.3520	0.3799	0.4152	0.4624	0.4933	0.5127	0.5176	0.4797	0.4102
	20	0.5500	0.7844	0.3725	0.3916	0.4814	0.5776	0.6638	0.7537	0.8094	0.8138	0.7502
	30	0.7106	0.9170	0.2804	0.4056	0.4982	0.5931	0.6927	0.8026	0.8979	0.9345	0.9289
	50	0.8734	0.9906	0.4242	0.4804	0.5168	0.5977	0.6983	0.8207	0.9296	0.9856	0.9949
	100	0.9889	1	0.4468	0.4451	0.4976	0.5757	0.6676	0.7875	0.9217	0.9929	1
3	10	0.2086	0.3987	0.3078	0.3294	0.3590	0.4090	0.4347	0.4504	0.4186	0.3689	0.2898
	20	0.3450	0.6690	0.3019	0.3221	0.3872	0.4788	0.5948	0.6963	0.7597	0.7496	0.6125
	30	0.4445	0.8352	0.1904	0.2985	0.3919	0.4918	0.6167	0.7414	0.8632	0.9215	0.8563
	50	0.6127	0.9634	0.3015	0.3528	0.3870	0.4470	0.5638	0.7133	0.8925	0.9807	0.9917
	100	0.8655	0.9993	0.3325	0.3371	0.3523	0.4211	0.4860	0.6255	0.8054	0.9732	0.9999

Table 13. Power comparisons of K, T and C_w for the directional scenario, $b = 6$.

σ	n	K	T	$C_{0.1}$	$C_{0.2}$	$C_{0.3}$	$C_{0.4}$	$C_{0.5}$	$C_{0.6}$	$C_{0.7}$	$C_{0.8}$	$C_{0.9}$
0.5	10	0.8733	0.8486	0.6207	0.6741	0.7234	0.7687	0.8044	0.8270	0.8529	0.8622	0.8773
	20	0.9902	0.9788	0.7556	0.7958	0.8432	0.8880	0.9151	0.9414	0.9661	0.9793	0.9884
	30	0.9989	0.9969	0.7828	0.8367	0.8861	0.9249	0.9514	0.9756	0.9870	0.9954	0.9992
	50	1	1	0.8427	0.9032	0.9306	0.9590	0.9808	0.9906	0.9965	0.9997	0.9999
	100	1	1	0.9260	0.9421	0.9623	0.9808	0.9925	0.9974	0.9997	1	1
1	10	0.7040	0.7473	0.5125	0.5470	0.5954	0.6512	0.6902	0.7258	0.7345	0.7487	0.7372
	20	0.9257	0.9422	0.6343	0.6805	0.7439	0.7968	0.8507	0.8871	0.9295	0.9424	0.9454
	30	0.9799	0.9890	0.6570	0.7256	0.7896	0.8511	0.9031	0.9361	0.9691	0.9836	0.9902
	50	0.9992	0.9996	0.6789	0.7913	0.8432	0.8970	0.9418	0.9716	0.9911	0.9977	0.9994
	100	1	1	0.8272	0.8422	0.8759	0.9355	0.9645	0.9897	0.9977	0.9995	1
2	10	0.3980	0.5674	0.3923	0.4450	0.4912	0.5379	0.5820	0.6016	0.5928	0.5548	0.4806
	20	0.6320	0.8496	0.4231	0.5102	0.5830	0.6849	0.7630	0.8332	0.8766	0.8826	0.8186
	30	0.7762	0.9538	0.4525	0.4825	0.5972	0.6991	0.7971	0.8937	0.9507	0.9700	0.9531
	50	0.9211	0.9966	0.3565	0.4924	0.5949	0.6839	0.7932	0.9036	0.9755	0.9967	0.9987
	100	0.9963	1	0.5609	0.5590	0.5747	0.6576	0.7580	0.8739	0.9659	0.9992	1
3	10	0.2405	0.4594	0.3602	0.4012	0.4451	0.4929	0.5333	0.5286	0.4991	0.4242	0.3400
	20	0.3862	0.7506	0.3278	0.4217	0.5110	0.6081	0.7243	0.8129	0.8592	0.8318	0.6860
	30	0.5158	0.9004	0.3584	0.3837	0.4698	0.5855	0.7325	0.8636	0.9461	0.9635	0.9150
	50	0.6982	0.9867	0.2257	0.3482	0.4456	0.5377	0.6710	0.8339	0.9594	0.9965	0.9966
	100	0.9070	0.9998	0.4353	0.4328	0.4236	0.4709	0.5729	0.7352	0.9008	0.996	1

Table 14. Rankings of four investment projects by three financial experts

Case A					Case B				
	Investment Projects					Investment Projects			
Experts	P_1	P_2	P_3	P_4	Experts	P_1	P_2	P_3	P_4
A	1	2	3	4	A	1	2	3	4
B	1	4	2	3	B	3	1	2	4
C	1	3	4	2	C	2	3	1	4

first show the effect of the weights on the data. Table 17 represents the data after the transformation for $w = 0.8$ and $w = 0.3$. For example, the entry in the second row and the second column in Table 17 is obtained as $0.8^7 = 0.2097$, where 7 is the corresponding value from Table 16. Finally, Table 18 presents the weighted rank coefficient of concordance C_w for different values of w along with the chi-square values $b(n-1)C_w$, the values $F_{C_w} = (b-1)C_w/(1-C_w)$ and their corresponding

Table 15. The weighted rank coefficient of concordance C_w for both cases in Table 14

		$C_{0.1}$	$C_{0.2}$	$C_{0.3}$	$C_{0.4}$	$C_{0.5}$	$C_{0.6}$	$C_{0.7}$	$C_{0.8}$	$C_{0.9}$
Case A	Statistic	0.991	0.968	0.932	0.887	0.838	0.786	0.736	0.687	0.642
	p-value	0.062	0.049	0.059	0.062	0.059	0.068	0.080	0.094	0.122
Case B	Statistic	0.146	0.189	0.239	0.294	0.351	0.407	0.461	0.511	0.558
	p-value	0.635	0.635	0.639	0.514	0.391	0.318	0.255	0.207	0.182

Table 16. Seven input variables ranked by six different measures

	Input variables						
Measure	A	B	C	D	E	F	G
SRC	1	2	3	4	5	6	7
SRRC	1	7	2	3	4	5	6
PD	1	6	7	2	3	4	5
CV	1	5	6	7	2	3	4
PCC	1	4	5	6	7	2	3
PRCC	1	3	4	5	6	7	2

Table 17. Seven input variables ranked by six different measures

	Input variables						
Measure	A	B	C	D	E	F	G
				$w = 0.8$			
SRC	0.8	0.6400	0.5120	0.4096	0.3277	0.2621	0.2097
SRRC	0.8	0.2097	0.6400	0.5120	0.4096	0.3277	0.2621
PD	0.8	0.2621	0.2097	0.6400	0.5120	0.4096	0.3277
CV	0.8	0.3277	0.2621	0.2097	0.6400	0.5120	0.4096
PCC	0.8	0.4096	0.3277	0.2621	0.2097	0.6400	0.5120
PRCC	0.8	0.5120	0.4096	0.3277	0.2621	0.2097	0.6400
				$w = 0.3$			
SRC	0.3	0.0900	0.0270	0.0081	0.0024	0.0007	0.0002
SRRC	0.3	0.0002	0.0900	0.0270	0.0081	0.0024	0.0007
PD	0.3	0.0007	0.0002	0.0900	0.0270	0.0081	0.0024
CV	0.3	0.0024	0.0007	0.0002	0.0900	0.0270	0.0081
PCC	0.3	0.0081	0.0024	0.0007	0.0002	0.0900	0.0270
PRCC	0.3	0.0270	0.0081	0.0024	0.0007	0.0002	0.0900

Table 18. The weighted rank coefficient of concordance C_w for Example 6.2

	$C_{0.1}$	$C_{0.2}$	$C_{0.3}$	$C_{0.4}$	$C_{0.5}$	$C_{0.6}$	$C_{0.7}$	$C_{0.8}$	$C_{0.9}$
Statistic	0.990	0.962	0.915	0.853	0.777	0.693	0.605	0.521	0.443
χ^2-value	35.65	34.63	32.95	30.70	27.98	24.94	21.80	18.74	15.94
p-value	3.2e-06	5.1e-06	1.1e-05	2.9e-05	0.0001	0.0003	0.0013	0.0046	0.0141
F-value	513.27	125.95	54.08	28.97	17.44	11.28	7.67	5.43	3.97
p-value	0.0e+00	0.0e+00	9.4e-15	3.3e-11	1.4e-08	1.4e-06	4.7e-05	0.0007	0.0048

p-values. From this table, we can see that there is a strong agreement among the six measures, where the strength of evidence varies depending on the chosen value of w.

Example 6.3 In this example a set of monthly data of four market indices is used to illustrate the new weighted rank coefficient of concordance C_w, these indices are the Standard and Poor (S&P), the Financial Times (FT), the Nikkei (Nik), and the DAX index. Meintanis and Iliopoulos [14] used this data set to test the independence of the four indices as well as of all combinations of three or two of them. The sampling period was September 2001-December 2005, yielding a sample size of $n = 50$ filtered returns (filtered by Meintanis and Iliopoulos [14] using ARMA(1,1) process). Meintanis and Iliopoulos [14] found that these four indices and all com-

Table 19. Summary of the finance market indices

	Min.	Q_1	Median	Mean	Q_3	Max.
S&P	-0.0970	-0.0215	-0.0015	0.0002	0.0261	0.0724
FT	-0.0788	-0.0166	0.0045	0.0000	0.0159	0.0580
Nikkei	-0.0981	-0.0303	-0.0063	-0.0002	0.0264	0.1513
DAX	-0.0842	-0.0387	-0.0029	0.0003	0.0245	0.1385

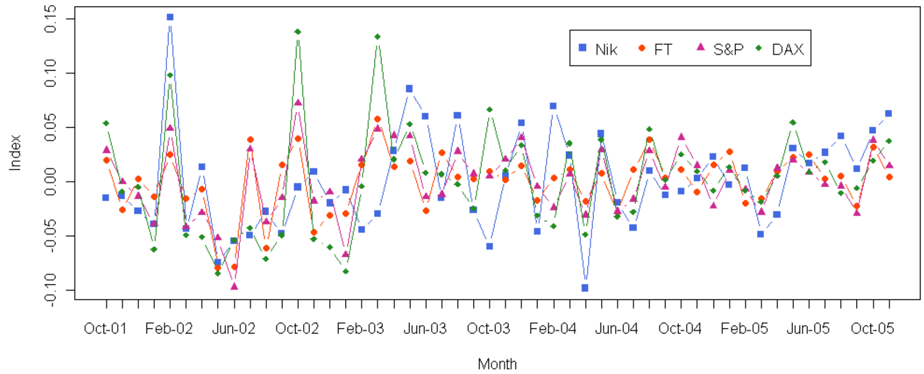

Figure 3. The original data of the four market indices

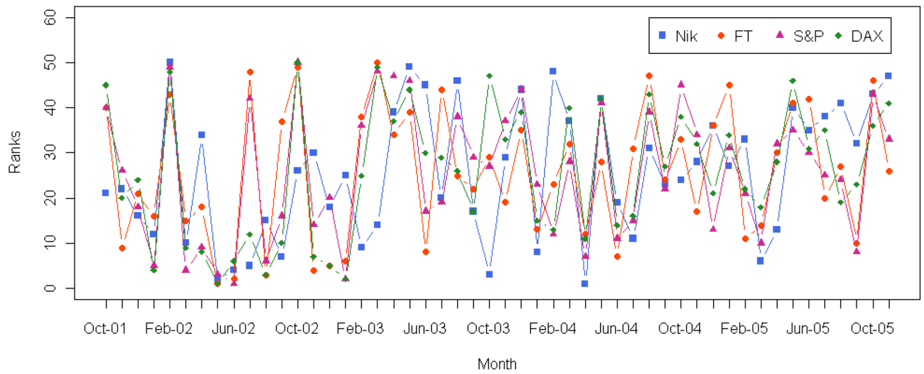

Figure 4. The rank data of the four market indices

binations of three of them are highly dependent, where the most dependent triple was found to be (S&P, DAX, FT) and the least dependent (S&P, FT, Nik). Moreover, all pairs of indices are also highly dependent, except for (FT, Nik), as the most dependent pair was (S&P, DAX) and the least dependent pair was (FT, Nik) followed by (S&P, Nik). Table 19 gives six summaries of these indices, namely their minimum, first quartile, median, mean, third quartile and maximum.

Figures 3, 4 and 5 show the original data of the four indices (after ARMA(1,1) filtering), the ranks of these data and the weighted ranks data with $w = 0.9$, respectively. Since these indices are all calculated differently it make sense in order to compare them to use the ranks instead of the data itself. Notice that higher values (ranks) in Figure 4 are transformed to lower values (weighted ranks) in Figure 5 and visa versa, e.g. rank 1 is transformed to 0.9 when using the weight $w = 0.9$.

Table 20 gives the values of K, $\chi^2_K = b(n-1)K$, T and $\chi^2_T = b(n-1)T$ and the corresponding p-values, and the values of C_w at different values of $w = 0.1(1)0.9$ along with $\chi^2_{C_w} = b(n-1)C_w$ and the corresponding p-values. We consider three cases. In Case I we give the smallest observation the rank 1 to the largest observation the rank 50. In Case II we do the reverse, namely we give the largest observation the rank 1 to the smallest observation the rank 50. Finally in Case III

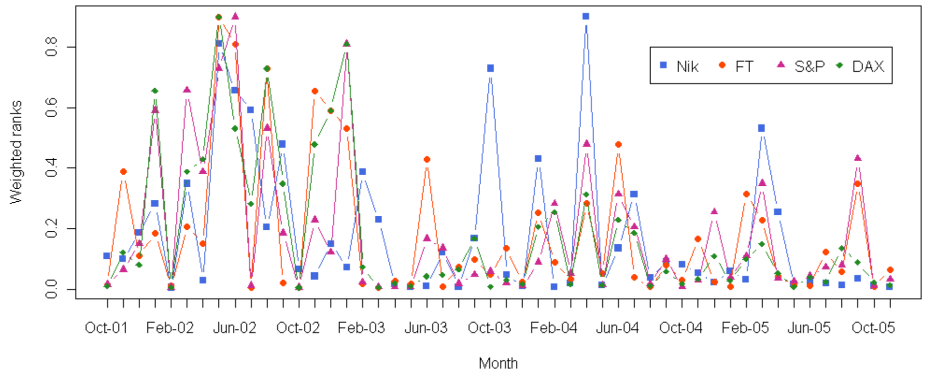

Figure 5. The weighted rank data with $w = 0.9$ of the four market indices

Table 20. Concordance coefficients of the four market indices

	Case I			Case II			Case III		
	value	χ^2 value	p-value	value	χ^2 value	p-value	value	χ^2 value	p-value
K	0.675	132.28	1.39e-09	0.675	132.28	1.39e-09	0.428	83.95	0.0014
T	0.705	138.19	1.97e-10	0.636	124.62	1.62e-08	0.336	65.82	0.0546
$C_{0.1}$	0.400	78.44	4.78e-03	0.412	80.67	2.93e-03	0.189	37.01	0.8960
$C_{0.2}$	0.442	86.64	7.35e-04	0.461	90.44	2.90e-04	0.202	39.60	0.8290
$C_{0.3}$	0.486	95.34	8.22e-05	0.509	99.76	2.53e-05	0.215	42.13	0.7460
$C_{0.4}$	0.532	104.25	7.28e-06	0.551	107.92	2.55e-06	0.227	44.45	0.6580
$C_{0.5}$	0.577	113.01	5.74e-07	0.582	114.05	4.20e-07	0.239	46.78	0.5640
$C_{0.6}$	0.618	121.18	4.77e-08	0.598	117.22	1.61e-07	0.254	49.73	0.4440
$C_{0.7}$	0.653	127.94	5.63e-09	0.597	116.92	1.77e-07	0.276	54.11	0.2860
$C_{0.8}$	0.675	132.25	1.40e-09	0.584	114.48	3.70e-07	0.307	60.22	0.1310
$C_{0.9}$	0.688	134.87	5.94e-10	0.598	117.12	1.66e-07	0.348	68.28	0.0356

Table 21. Concordance coefficients of all 3 combinations of the market indices

	(S&P, FT, Nik)	(S&P, FT, DAX)	(S&P, Nik, DAX)	(FT, Nik, DAX)
K	0.645**	0.836**	0.715**	0.649**
T	0.685**	0.854**	0.717**	0.695**
$C_{0.1}$	0.364	0.571**	0.344	0.587**
$C_{0.2}$	0.413	0.603**	0.378	0.621**
$C_{0.3}$	0.466*	0.640**	0.421	0.647**
$C_{0.4}$	0.521**	0.679**	0.470*	0.665**
$C_{0.5}$	0.575**	0.720**	0.525**	0.675**
$C_{0.6}$	0.623**	0.761**	0.579**	0.679**
$C_{0.7}$	0.658**	0.801**	0.630**	0.677**
$C_{0.8}$	0.669**	0.831**	0.672**	0.671**
$C_{0.9}$	0.664**	0.845**	0.711**	0.670**

'*' significant at 5% and '**' significant at 1%

we rank the absolute difference of the observations from the median for the corresponding index, so the smallest absolute difference takes the rank 1 to the largest absolute difference which takes the rank 50. From Table 20 we see that for Cases I and II all concordance coefficients lead to rejection of the null hypothesis at significance level 1%, so there is an agreement between the four market indices, both when the ranks of the largest positive or negative values are considered. For Case III we do not reject the null hypothesis of no agreement for all concordance coefficients, except for K and $C_{0.9}$ where we reject the null hypothesis at significance level 1% and 5%, repectively.

Table 21 presents the same statistics for all combinations of three market indices. These show significant agreement using all concordance coefficients T, K and C_w except for the triple (S&P, FT, Nik) with $w = 0.1, 0.2$ and for the triple (S&P, Nik, DX) with $w = 0.1, 0.2, 0.3$. Figures 6 and 7 show the original data (after filtering), the ranks of these data and the weighted ranks data with $w = 0.9$, for the most

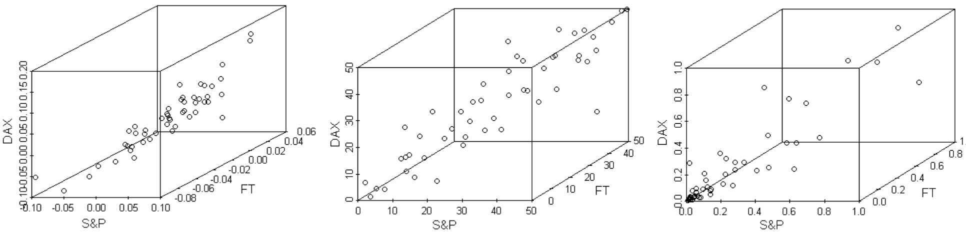

Figure 6. The original data of the triple (S&P, FT, DAX), the ranks, and the weighted ranks with $w = 0.9$

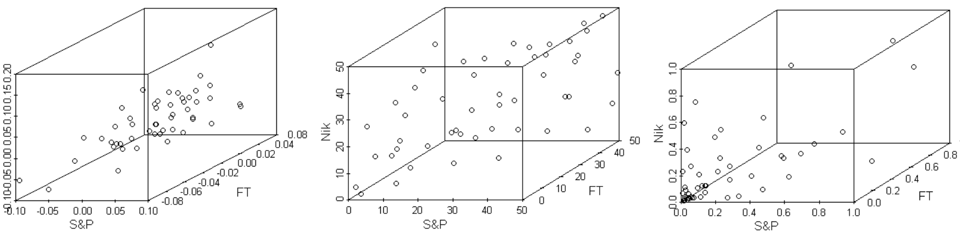

Figure 7. The original data of the triple (S&P, FT, Nik), the ranks, and the weighted ranks with $w = 0.9$

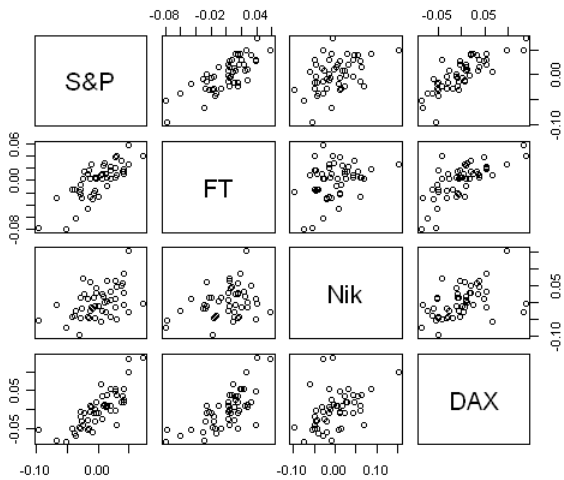

Figure 8. Scatter plot of all possible pairs of the market indices for the original data

Table 22. Rank correlation coefficients for all pairs of the market indices

	(S&P, FT)	(S&P, Nik)	(S&P, DAX)	(FT, Nik)	(FT, DAX)	(Nik, DAX)
r_s	0.742**	0.414**	0.824**	0.245*	0.696**	0.478**
r_T	0.751**	0.459**	0.801**	0.372**	0.792**	0.465**
$R_{0.1}$	0.087	-0.023	-0.005	0.077	0.990**	0.076
$R_{0.2}$	0.207	-0.015	0.049	0.169	0.960**	0.167
$R_{0.3}$	0.331*	0.013	0.134	0.253*	0.916**	0.245*
$R_{0.4}$	0.450**	0.067	0.245*	0.326*	0.863**	0.304*
$R_{0.5}$	0.553**	0.148	0.373**	0.385**	0.814**	0.339**
$R_{0.6}$	0.634**	0.248*	0.510**	0.423**	0.783**	0.348**
$R_{0.7}$	0.681**	0.350**	0.645**	0.428**	0.780**	0.338**
$R_{0.8}$	0.695**	0.428**	0.756**	0.389**	0.790**	0.341**
$R_{0.9}$	0.714**	0.453**	0.821**	0.322*	0.769**	0.426**

'*' significance at 5% and '**' significance at 1%

and least dependent triples according to Meintanis and Iliopoulos [14], i.e. (S&P, FT, DAX) and (S&P, FT, Nik), respectively. We can see from the third plot in Figures 6 and 7, how the transformation, using the weighted ranks, leads to focus on the top ranks.

Now let us consider the 6 possible pairwise comparisons of these market indices,

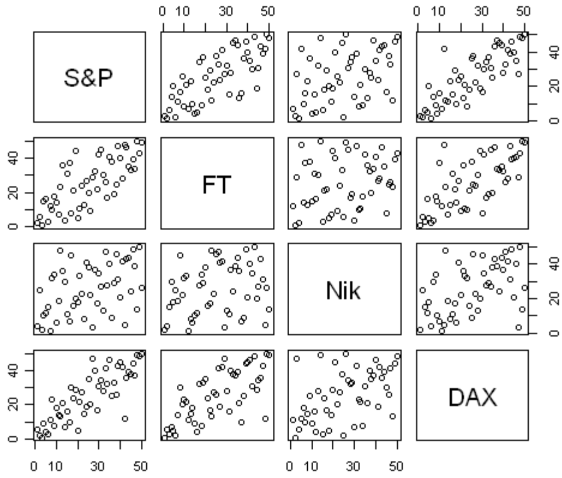

Figure 9. Scatter plot of all possible pairs of the market indices for the ranked data

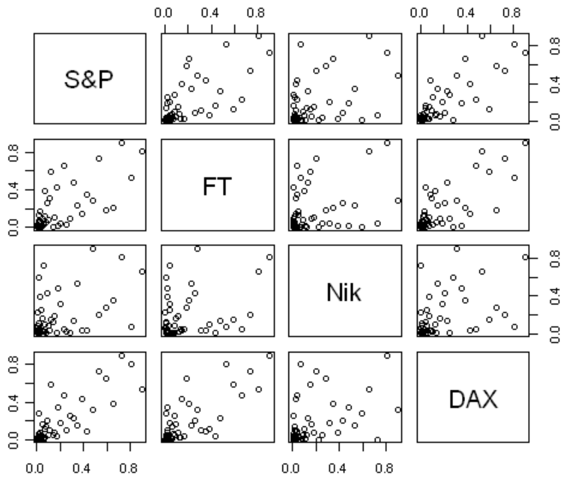

Figure 10. Scatter plot of all possible pairs of the market indices for the weighted ranked data, where $w = 0.9$

(S&P, FT), (S&P, Nik), (S&P, DAX), (FT, Nik), (FT, DAX) and (Nik, DAX). Figures 8, 9 and 10 show the matrix of all possible pairs of market indices for the original data (after filtering), the ranks of these data and the weighted ranks data with $w = 0.9$, respectively. We can see from these figures that the pair (S&P, DAX) (respectively, (FT,Nik)) is more (respectively, less) in rank agreement, which coincides with the results obtained by Meintanis and Iliopoulos [14]. Table 22 presents the Spearman's rank correlation coefficient r_s, top-down rank correlation r_T [6] and the weighted rank correlation R_w in (3) at different values of w. In order to test the null hypothesis of independence against the alternative of a positive correlation, the critical value $z_{1-\alpha}/\sqrt{n-1}$ should be used [6, 8, 13], that is all the rank correlation coefficients will be compared with 0.235 and 0.332 for significance level 5% and 1%, respectively. Table 22 shows that for the pair (FT, DAX) we always reject the null hypothesis of independent, so there is evidence of positive correlation between these two indices. Spearman's rank correlation coefficient r_s and top-down rank correlation r_T always indicate positive correlation for all 6 pairs while for R_w this varies depending on the value of w. For example, we do not reject the null hypothesis of independence for the pairs (S&P, FT), (FT, Nik) and (Nik, DAX) for $w = 0.1$ and $w = 0.2$. In addition we do not reject the null hypothesis for the pair (S&P, Nik) (respectively, (S&P, DAX)) for any $w \leq 0.5$ (respectively,

$w \leq 0.3$). We can also notice that the results from Spearman's rank correlation coefficient r_s and top-down rank correlation r_T are close to the results from the weighted rank correlation $R_{0.9}$. Finally, the negative correlation values for the two pairs (S&P, Nik) and (S&P, DAX), using the weighted rank correlation R_w with very small values of w, indicate some disagreement between these indices in the top rankings, yet these values are very small and close to zero, so there is no statistical evidence that there is a negative correlation between these indices.

7. Conclusion

In this paper we have presented a new weighted rank coefficient of concordance when there are $b > 2$ independent sources of rankings and the focus is on agreement of the top rankings. We also presented the limiting distribution of this weighted rank coefficient of concordance under the null hypothesis of no agreement between the rankings. We also carried out an extensive simulation study to compare the performance of the proposed weighted concordance coefficient with Kendall's and the top-down concordance coefficients. The simulation study showed that the proposed weighted concordance coefficient performs very well in the directional rank agreement scenario especially for large values of the weight w. We illustrated the use of the new weighted rank coefficient of concordance via examples including an example of financial markets indices. There is no particular criterion for the choice of the weight w, its influence depends on the number of objects and to which extend one wants to focus on the top rankings.

Appendix

Proof [Proof of the relationship given in (8):] Let R_w^{av} be the mean value of the weighted rank correlation coefficients between the $\binom{b}{2}$ possible pairs, then from (4),

$$
\begin{aligned}
R_w^{av} &= \frac{2}{b(b-1)} \sum_{j<k} \left\{ \frac{\sum_{i=1}^{n} \left(w^{R_{ji}} - (\sqrt{a_1}/n)\right)\left(w^{R_{ki}} - (\sqrt{a_1}/n)\right)}{a_2 - (a_1/n)} \right\} \\
&= \frac{2}{b(b-1)(a_2 - (a_1/n))} \sum_{i=1}^{n} \left\{ \sum_{j<k} \left(w^{R_{ji}} - \frac{\sqrt{a_1}}{n} \right) \left(w^{R_{ki}} - \frac{\sqrt{a_1}}{n} \right) \right\} \\
&= \frac{1}{b(b-1)(a_2 - (a_1/n))} \sum_{i=1}^{n} \left\{ \left[\sum_{j=1}^{b} \left(w^{R_{ji}} - \frac{\sqrt{a_1}}{n} \right) \right]^2 - \sum_{j=1}^{b} \left(w^{R_{ji}} - \frac{\sqrt{a_1}}{n} \right)^2 \right\} \\
&= \frac{1}{b(b-1)(a_2 - (a_1/n))} \left[b^2 (a_2 - (a_1/n))\, C_w - b\,(a_2 - (a_1/n)) \right] \\
&= \frac{b\, C_w - 1}{b - 1}
\end{aligned}
$$

■

Proof [Proof of Theorem 4.1:] The general form of the rank statistic for random blocks, Q_b, and its limiting distribution is given by the following theorem.

Theorem .1 *[4, p.173]*
The number of observations in each block will be fixed and denoted by n. The scores

$a(i)$, $1 \leq i \leq n$, will also be fixed and arbitrary but not constant. For $b \to \infty$, the number of blocks b will tend to ∞, the statistic

$$Q_b = \frac{n-1}{b}\left[\sum_{i=1}^{n}(a(i)-\bar{a})^2\right]^{-1}\sum_{i=1}^{n}\left(\sum_{j=1}^{b}a(R_{ji}) - b\,\bar{a}\right)^2 \qquad (1)$$

has asymptotically the χ^2-distribution with $n-1$ degrees of freedom.

From (5)-(7), the weighted rank concordance coefficient C_w can be written as a function of the general form of the rank statistic for random blocks in (1), as

$$C_w = \left[\sum_{i=1}^{n}\left(bw^i - \frac{b\sqrt{a_1}}{n}\right)^2\right]^{-1}\sum_{i=1}^{n}\left(w_{\cdot i} - \frac{b\sqrt{a_1}}{n}\right)^2 = \frac{Q_b}{b(n-1)}$$

where $a(i) = w^i$, $\sum_{i=1}^{n} a(i) = \sum_{i=1}^{n} w^i = a_2$, $\sum_{j=1}^{b} a(R_{ji}) = \sum_{j=1}^{b} w^{R_{ji}} = w_{.i}$ and $\bar{a} = \sqrt{a_1}/n$.

■

Proof [Proof of formula (9):] This is straightforward using the F statistic from the classical analysis of variance table, where the source of variation (SST) can be partitioned into three components; between rows (SSR), between columns (SSC) and errors (SSE). Then,

$$SST = \sum_{i=1}^{n}\sum_{j=1}^{b}\left(w^{R_{ji}} - \frac{\sqrt{a_1}}{n}\right)^2 = \sum_{i=1}^{n}\sum_{j=1}^{b} w^{2R_{ji}} - \frac{b\,a_1}{n}$$

$$SSC = \sum_{i=1}^{n}\sum_{j=1}^{b}\left(\frac{w_{\cdot i}}{b} - \frac{\sqrt{a_1}}{n}\right)^2 = \frac{1}{b}\sum_{i=1}^{n} w_{\cdot i}^2 - \frac{b\,a_1}{n}$$

$$SSE = SST - SSC = \sum_{j=1}^{b}\sum_{i=1}^{n}\left(w^{R_{ji}} - \frac{w_{\cdot i}}{b}\right)^2 = \sum_{i=1}^{n}\sum_{j=1}^{b} w^{2R_{ji}} - \frac{1}{b}\sum_{i=1}^{n} w_{\cdot i}^2$$

There is no variation between rows here, so $SSR = 0$, since the row sums are all equal. Then, the following statistic has F distribution with $n-1$ and $(b-1)(n-1)$ degrees of freedom,

$$F = \frac{MSC}{MSE} = \frac{(b-1)(n-1)SSC}{(n-1)SSE} = \frac{(b-1)C_w}{1-C_w}$$

The distribution of this statistic can be used as an approximation to the exact distribution of C_w, as shown by Iman and Davenport [7] for the Friedman test in the case of ties. Thus C_w can be written as

$$C_w = \frac{F}{(b-1)+F} = \frac{SSC}{SSC+SSE} = \frac{SSC}{SST}$$

■

Acknowledgement

I am grateful to Simos Meintanis, George Iliopoulos for providing the data used in Example 6.3.

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