A generalized system reliability model based on survival signature and multiple competing failure processes

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Abstract Degradation-based system reliability analysis has been extensively conducted, but the components in a system are assumed to experience similar degradation and shock processes, neglecting actual failure mechanisms. However, multiple types of components in a system may work under different operational conditions and break down due to different failure mechanisms. Hence, a new generalized reliability model is proposed for systems with arbitrary structures experiencing multiple degradation and shock processes, including pure degradation processes (DPs), independent and dependent competing failure processes (CFPs). In this work, the Tweedie exponential-dispersion (TED) process is utilized to describe multiple degradation processes of the components, which contains the Wiener, Gamma, inverse Gaussian, and other processes as special cases. Based on multiple DPs and CFPs, a generalized reliability model is established by utilizing the structure analysis method, the survival signature, which allows the proposed method to be applied to various structural systems. Finally, an example of an automotive braking system with four types of components experiencing multiple DPs and CFPs is applied to illustrate the proposed model.

Keywords: competing failure processes; degradation; Tweedie exponential-dispersion process; survival signature; system reliability.

1. Introduction

System reliability analysis has been extensively conducted to accurately evaluate the probability that the devices function normally under required operational conditions [1]. Research on the reliability of complex systems based on survival signature and the reliability of simple systems based on degradation and shock processes has been widely investigated [2-3]. However, the reliability analysis for systems with complex structures based on degradation-shock processes and survival signature is rarely conducted. Therefore, a new generalized reliability model is proposed for complex systems based on

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multiple competing failure processes (CFPs) and survival signature, which is applicable for systems with general structures, including series, parallel, bridge, and network structures. Also, the proposed model is suitable for systems experiencing multiple degradation and shock processes, including pure degradation processes (DPs), independent and dependent competing failure processes (CFPs).

Systems are mainly supposed to fail due to soft and hard failures [4]. Soft failures are caused by degradation, such as wear, corrosion, crack growth, and other degrading performance characteristics [5]. Hard failures are usually caused by random shocks, such as sudden loads, intense temperature fluctuations, and other abruptly changing environmental conditions [6]. Research on the reliability of systems exposed to DPs and CFPs has been widely carried out, in which the Wiener, Gamma, and inverse Gaussian processes are commonly used. For example, Gao et al. [7] established a reliability model for a two-unit system based on the Wiener process with consideration of the degradation interdependence. Liu et al. [8] proposed a reliability method for systems with uncertainty based on the Wiener process and four shock patterns. Li et al. [9] presented a new storage reliability method for missiles considering the dependence between the random failure process and the inverse Gaussian degradation process. Yousefi et al. [10] presented a new reliability model for dependent CFPs and provided the corresponding maintenance policies with conditional thresholds based on the Gamma processes. The Wiener, Gamma, and inverse Gaussian processes are suitable for describing stochastic processes with different characteristics, such as monotonic or nonmonotonic processes.

To describe various stochastic processes more generally, Tweedie [11] presented the Tweedie exponential-dispersion (TED) process, which includes the commonly used Wiener, Gamma, and inverse Gaussian processes as special cases. Many researchers modelled degradation processes and estimated system reliability with the TED process due to its generality. Yan et al. [12] modelled the performance degradation process of flax fibers by the TED process and completed the durability and reliability estimation. Yan et al. [13] proposed a reliability method for photovoltaic modules by modelling the dynamic and random degradation process with the TED process. Chen et al. [14] extended the application of the TED process by presenting a nonlinear TED process with the consideration of the random effects on degradation. In the above research, the reliability analysis for systems exposed to various DPs and CFPs has been extensively investigated using the TED process and other stochastic processes. However, the degradation of the components in a system is assumed to follow the same stochastic process with different parameters, and the DPs and the shock processes are considered to affect each other in the same way. The assumptions may not be acceptable for some systems with multiple types of components, which are designed for different functional purposes and work under different operational conditions. The components may break down due to different failure mechanisms under different DPs and CFPs. Therefore, in this work, a generalized reliability model is proposed based on the TED process for multi-component systems experiencing multiple DPs and CFPs.

To complete the system reliability analysis, the structure analysis needs to be conducted after completing the component reliability analysis. Typical structures include series [15], parallel [16], mixed series and parallel [17], bridge [18], and network structures [19]. For systems with simple structures, such as series, parallel, and series-parallel systems, the reliability has been widely analyzed based on DPs and CFPs. For example, Dong et al. [20] presented a new reliability model for mixed series and parallel systems with CFPs by considering the self-healing and aggravating effects on the random shocks. Kong et al. [21] proposed a generalized reliability model for mixed series and parallel systems based on multiple dependent DPs. Yousefi et al. [22] established a reliability model for series-parallel systems experiencing multiple CFPs, by dividing the components into groups according to their locations.

However, for multi-component systems with complex structures, such as the bridge and network systems, the structure analysis may be challenging to complete by considering alternative series and parallel subsystems.

To complete the reliability analysis for systems with general structures, the survival signature proposed by Coolen and Coolen-Maturi [23-24] is utilized, which is a structure analysis technique widely applicable to various kinds of systems. For example, Salomon et al. [25] developed a reliability model for complex systems with uncertainty based on survival signature and fuzzy probabilities. Qin and Coolen [26] proposed a reliability model for multi-state systems with multi-state components based on survival signature. Huang et al. [27] applied the survival signature to phased mission systems and established an efficient reliability model. Coolen-Maturi et al. [28] proposed a joint survival signature for multiple systems with shared components. Reed et al. [29] extended the application of survival signature to a K-terminal network system and analyzed the system reliability after completing the structure analysis. Different from the previous work based on survival signature, which considered the systems to fail due to degradation. Huang et al. [30] applied the survival signature for system reliability taking both soft and hard failures into consideration. Similarly, Hashemi et al. [31], and Tavangar and Hashemi [32] provided reliability analysis methods based on the survival signature, to derive maintenance policies for systems exposed to degradation and shocks. In the above research, reliability analysis for various systems has been extensively conducted based on survival signature, but the reliability is estimated by modelling time-to-failure data with lifetime distributions. However, for complex systems with multiple types of components, it is challenging to obtain enough failure data for all components, even with the help of accelerated life tests [33-34]. Therefore, a generalized reliability model is proposed for complex systems based on survival signature, in which the reliability analysis is completed by modelling the degradation data with the TED process.

In this paper, based on survival signature and the TED process, a generalized reliability model is proposed for systems with arbitrary structures experiencing multiple DPs and CFPs. The main contributions of this work are as follows:

(1) A generalized reliability model is proposed for complex systems with multiple types of components based on DPs, CFPs, and survival signature, instead of lifetime distributions requiring time-to-failure data. The proposed method is presented based on the TED process, which allows to avoid predetermining the DPs of components as specific ones, such as the Wiener, Gamma, and inverse Gaussian processes.

(2) By utilizing survival signature, a new degradation-based reliability model is developed for systems with arbitrary structures, including the series, parallel, series-parallel, bridge, and network systems.

(3) Instead of assuming the DPs and CFPs of multiple types of components to be similar, according to the practical failure mechanisms of the components, generalized reliability models are provided for complex systems experiencing multiple degradation and shock processes, including pure DPs, independent and dependent CFPs.

The remainder of the paper is organized as follows. In Section 2, the TED process is utilized to describe the DPs of components, and generalized component reliability models are developed based on the TED process when experiencing pure DPs, independent and dependent CFPs. Section 3 develops new reliability models based on survival signature for systems with arbitrary structures experiencing multiple DPs and CFPs. In Section 4, an illustrative example of an automotive braking system with four types of components is provided to show the computation processes and validity of the proposed methods. Conclusions and some future challenges are presented in Section 5

2. Reliability analysis for components based on DP and CFPs

In a practical engineering system, the performance characteristics of components may degrade differently. This section conducts the reliability analysis for components exposed to different degradation and shock processes, including the general TED process, extreme shock process, independent and dependent CFPs.

2.1 Reliability analysis for the TED degradation process

In this section, a general class of stochastic processes, the TED process, is utilized to model the degradation of multiple types of components. If the initial degradation value $X(0)$ is 0, the increments are independent and stationary, and the component degradation follows the TED process, then the PDF of the degradation value $X(t)$ can be obtained using the saddle-point approximation method [13-14].

$$f_{X(t)}\left(x|\mu,\lambda,p,t\right)=\sqrt{\frac{\lambda}{2\pi t^{1-p}x^{p}}}\exp\left\{-\frac{\lambda t}{2}d\left(\frac{x}{t};\mu\right)\right\} \tag{1}$$

where $p \in (-\infty, 0]\cup [1, +\infty)$, μ is the drift parameter, λ is the diffusion parameter, and

$$d\left(\frac{x}{t};\mu\right)=\begin{cases}2\left[\frac{x}{t}\ln\left(\frac{x}{\mu t}\right)-\frac{x}{t}+\mu\right], \text{ if } p=1\\ 2\left[\ln\left(\frac{\mu t}{x}\right)+\frac{x}{\mu t}-1\right], \text{ if } p=2\\ 2\left[\frac{x^{2-p}}{t^{2-p}(1-p)(2-p)}-\frac{X\mu^{(1-p)}}{t(1-p)}+\frac{\mu^{(2-p)}}{(2-p)}\right], \text{ for } p\neq 1,2\end{cases}$$

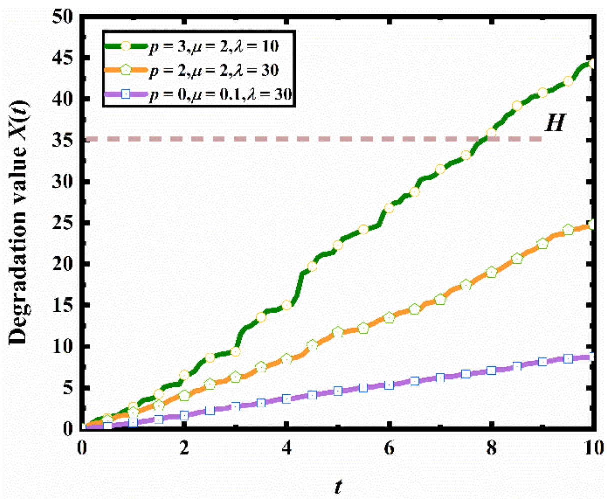

Fig. 1 TED degradation process

We denote the TED process by $X(t) \sim TED\ (\mu t, \lambda, p)$, the mean is $E[X(t)] = \mu t$, and the variance is $Var[X(t)] = \mu^{P}t/\lambda$. The TED processes with different values of parameters are shown in **Fig. 1**. The soft failure is triggered when the degradation value $X(t)$ exceeds the failure threshold H. Based on the generalized TED process, the reliability of components experiencing pure DPs can be expressed as:

$$\begin{aligned} R_S(t) &= P\left(X(t) < H\right) \\ &= \int_0^H f_{X(t)}\left(x \middle| \mu, \lambda, p, t\right) dx \\ &= \int_0^H \sqrt{\frac{\lambda}{2\pi t^{1-p} x^p}} \exp\left\{-\frac{\lambda t}{2} d\left(\frac{x}{t}; \mu\right)\right\} dx \end{aligned} \tag{2}$$

where H is the failure threshold of the degradation process

2.2 Reliability analysis for the extreme shock process

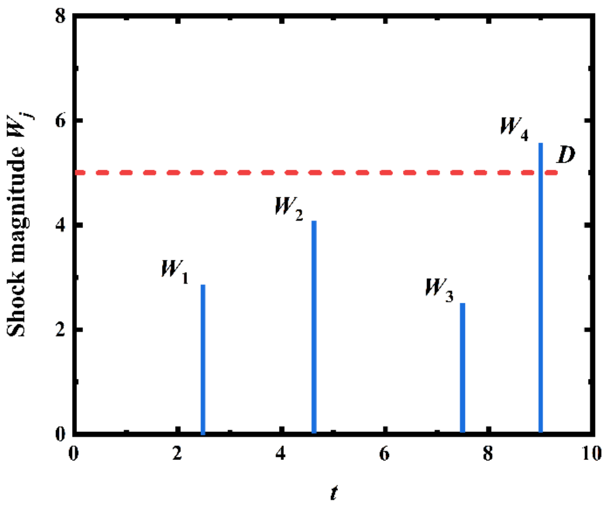

Fig. 2 The extreme shock process

The extreme shock process [5-6] is widely utilized to model the random stress on components, such as random thermal shocks and vibrations. As shown in **Fig. 2**, a hard failure is triggered when the magnitudes of the random shocks exceed the failure threshold D. The arrival of shocks is modeled as a homogeneous Poisson process with rate θ. The probability that $N(t)$ shocks occur in (0, t] can be derived as:

$$P\left(N(t) = n\right) = \exp\left(-\theta t\right) \cdot \frac{\left(\theta t\right)^n}{n!} \tag{3}$$

where $N(t)$ is the number of shocks, n = 0, 1, 2, …, θ is the arrival rate of the shock process. The magnitudes of random shocks are assumed to be independent identically distributed with a normal distribution, denoted as $W_j \sim N(\mu_W, \sigma_W^2)$. The probability that the components survive a random shock can be expressed as:

$$P\left(W_j < D\right) = \Phi\left(\frac{D - \mu_W}{\sigma_W}\right) \tag{4}$$

where D is the shock failure threshold and $\Phi(\cdot)$ is the cumulative distribution function (CDF) of a standard normal random variable. The reliability of components suffering from hard failures caused by the random shock process can be derived as:

$$\begin{aligned} R_H(t) &= \sum_{n=0}^{\infty} P\left(N(t)=n, \bigcap_{j=1}^{n}\left(W_j < D\right)\right) \\ &= \sum_{n=0}^{\infty} P\left(N(t)=n\right) \cdot \prod_{j=1}^{n} P\left(W_j < D\right) \\ &= \sum_{n=0}^{\infty} \exp(-\theta t) \frac{(\theta t)^n}{n!} \left[\Phi\left(\frac{D-\mu_W}{\sigma_W}\right)\right]^n \end{aligned} \tag{5}$$

To simplify the expressions of the equations in this work, we denote exp(-θt) by $r(t)$, denote(-θt)^n /n! by $r_1(t, n)$, and denote $[\Phi((D-\mu_W)/\sigma_W)]^n$ by $r_2(W_j|D, n)$, then Eq. (5) can be expressed as:

$$R_H(t) = \sum_{n=0}^{\infty} r(t) r_1(t,n) r_2(W_j | D, n) \tag{6}$$

2.3 Reliability analysis for independent CFPs

As shown in **Fig. 3**, when experiencing degradation, some components may also suffer from random shocks. The soft failure caused by degradation and the hard failure caused by random shocks are considered to be independent CFPs. No matter which one happens, it can lead the components to fail, but the degradation and shock processes are assumed not to affect each other. For example, a brake pad fails if the cumulated wear reaches the failure threshold H or the friction coefficient drops suddenly due to thermal shocks [35-36], the wear and the change of friction coefficient are assumed to be independent.

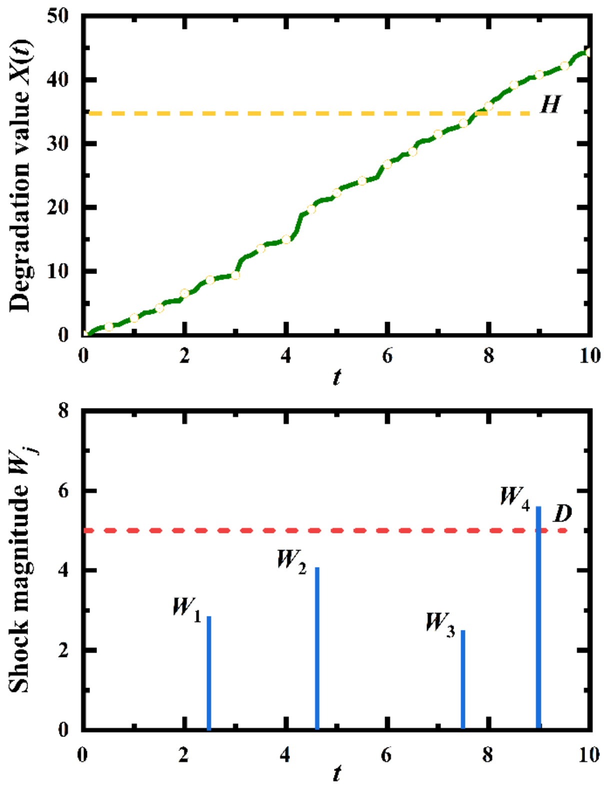

Fig. 3 Description of the independent degradation-shock processes

If the degradation and shock processes of a component can be modelled by the generalized TED process and the extreme shock process, then the reliability of a component experiencing the independent CFPs can be expressed as:

$$
\begin{aligned}
R_{\text{ind}}(t) &= \sum_{n=0}^{\infty} P\left(X(t)<H, N(t)=n, \bigcap_{j=1}^{n}\left(W_j<D\right)\right) \\
&= \sum_{n=0}^{\infty} P\left(X(t)<H\right) P\left(N(t)=n\right) \prod_{j=1}^{n} P\left(W_j<D\right) \\
&= \sum_{n=0}^{\infty} P\left(X(t)<H\right) P\left(N(t)=n\right) P\left(W_j<D\right)^n \\
&= \sum_{n=0}^{\infty} \int_0^H f_{X(t)}\left(x \middle| \mu, \lambda, p, t\right) dx \cdot r(t) r_1(t,n) r_2(W_j | D, n)
\end{aligned} \tag{7}
$$

2.4 Reliability analysis for dependent CFPs with changing hard failure threshold

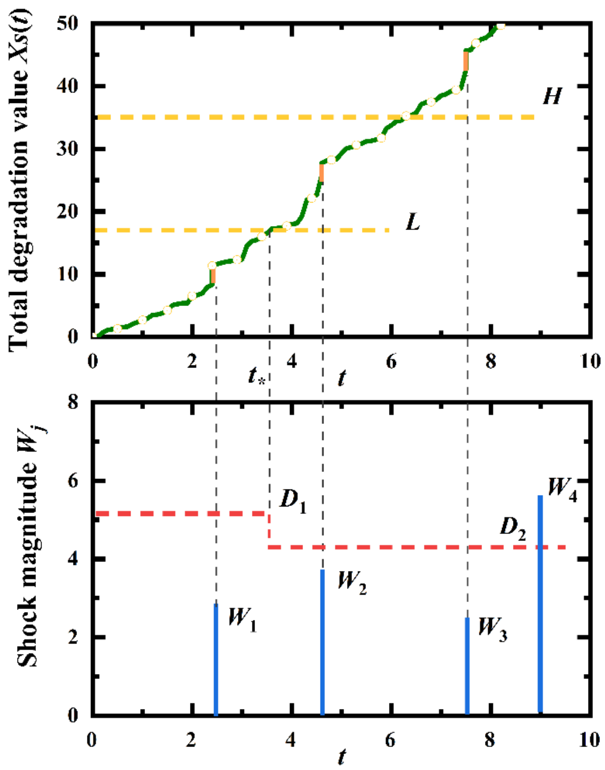

Fig. 4 Description of the dependent degradation-shock processes

As shown in **Fig. 4**, some components may fail due to dependent CFPs. One is the degradation process, which is the sum of the natural degradation (the green parts in **Fig. 4**) and the sudden increments (the orange parts in **Fig. 4**) caused by random shocks. The other is a shock process with changing failure thresholds, in which the resistance ability of the component to random shocks is considered to decline with increasing degradation levels. When the total degradation value $Xs(t)$ exceeds a certain level H, then the component fails due to a soft failure. When the magnitude of a shock exceeds the corresponding failure threshold (D_1 or D_2), then the component fails due to a hard failure. The two failure processes are dependent and competing, the occurrence of either one can lead the component to fail. For example, a handbrake cable needs to be replaced when the plastic deformation cumulates to the failure threshold

H or when the cable snaps under sudden loads W_j [37], where j = 1, 2, 3, …. The capacity of the handbrake cable to avoid snapping declines from D_1 to D_2 when the plastic deformation increases to L at time point t_*.

Assume the natural degradation of components follows the generalized TED process, that is, $X(t) \sim$ TED (μt, λ, p). The shock process is considered to be modeled by a homogeneous Poisson process with arrival rate θ, and the shock damage is assumed to be linear with the shock magnitude, which can be expressed as follows.

$$Z_j = aW_j \tag{7}$$

where Z_j is the damage caused by the jth shock, $Z_j \sim N(\mu_Z, \sigma_Z^2)$, the mean of the damages is $\mu_Z = a\mu_W$ and its variance is $\sigma_Z^2 = a^2\sigma_W^2$, where a is a constant. The total degradation caused by random shocks is $S(t)$, which can be derived as:

$$S(t) = \begin{cases} \sum_{j=1}^{N(t)} Z_j, & \text{if } N(t) \neq 0 \\ 0, & \text{if } N(t) = 0 \end{cases} \tag{9}$$

The PDF of the degradation caused by the shocks is:

$$g_{S(t)}\left(u \middle| N(t) = n, n \neq 0\right) = \frac{1}{\sqrt{2\pi n\sigma_Z^2}} \exp\left[-\frac{(u - n\mu_Z)^2}{2n\sigma_Z^2}\right] \tag{10}$$

The total degradation of a component, $Xs(t)$, is the sum of the natural degradation and sudden degradation increments caused by random shocks.

$$Xs(t) = X(t) + S(t) \tag{11}$$

If the total degradation cumulates to H, then the component fails due to a soft failure. The probability that the component survives from a soft failure is:

$$\begin{aligned} P_S(t) &= \sum_{n=0}^{\infty} P\left(Xs(t) < H, N(t) = n\right) \\ &= \sum_{n=0}^{\infty} P\left(X(t) + S(t) < H \middle| N(t) = n\right) P\left(N(t) = n\right) \\ &= P\left(X(t) < H \middle| N(t) = 0\right) P\left(N(t) = 0\right) \\ &+ \sum_{n=1}^{\infty} \int_0^H P\left(X(t) < H - u \middle| S(t) = u, N(t) = n\right) g_{S(t)}\left(u \middle| N(t) = n\right) du \cdot P\left(N(t) = n\right) \\ &= \int_0^H \left[f_{X(t)}\left(x \middle| \mu, \lambda, p, t\right) dx \right] \cdot r(t) \\ &+ \sum_{n=1}^{\infty} \int_0^H \int_0^{H-u} \left[f_{X(t)}\left(x \middle| \mu, \lambda, p, t\right) g_{S(t)}\left(u \middle| N(t) = n\right) dx du \right] \cdot r(t) r_1(t, n) \end{aligned} \tag{12}$$

If the magnitude of a shock exceeds the hard failure threshold, the component fails due to a hard failure. When the component deteriorates, its capacity to resist random shocks gets weaker, making it more vulnerable to breaking down. For example, brake oil gradually gets contaminated due to chemical reactions and moisture absorption during brake operation. Oil contamination can reduce the fluid boiling point and the pressure transmission, which makes the braking system more vulnerable to failures when experiencing thermal shocks [38]. Therefore, in this subsection, the hard failure threshold is considered to shift with the degradation levels. The probability that a component survives a hard failure at time t is:

$$
\begin{aligned}
P_H(t) &= \sum_{n=0}^{\infty}\sum_{n_1=0}^{n} P\left(N(t_*)=n_1, N(t-t_*)=n-n_1, \bigcap_{j=1}^{n_1}\left(W_j<D_1\right), \bigcap_{j=n_1+1}^{n}\left(W_j<D_2\right)\right) \\
&= \sum_{n=0}^{\infty}\sum_{n_1=0}^{n} P\left(N(t_*)=n_1\right) P\left(N(t-t_*)=n-n_1\right) \prod_{j=1}^{n_1} P\left(W_j<D_1\right) \prod_{j=n_1+1}^{n} P\left(W_j<D_2\right) \\
&= \sum_{n=0}^{\infty}\sum_{n_1=0}^{n} P\left(N(t_*)=n_1\right) P\left(N(t-t_*)=n-n_1\right) P\left(W_j<D_1\right)^{n_1} P\left(W_j<D_2\right)^{n-n_1} \\
&= \sum_{n=0}^{\infty}\sum_{n_1=0}^{n} r(t_*) r_1(t_*, n_1) r(t-t_*) r_1(t-t_*, n-n_1) r_2(W_j \mid D_1, n_1) r_2(W_j \mid D_2, n-n_1)
\end{aligned}
\tag{13}
$$

The reliability of a component experiencing dependent CFPs with changing hard failure thresholds can be calculated in the following two independent and mutually exclusive situations.

$$
\begin{aligned}
R_{\text{dep}}(t) &= P\left(A \mid B_1\right) P\left(B_1\right) + P\left(A \mid B_2\right) P\left(B_2\right) \\
&= P\left(AB_1\right) + P\left(AB_2\right)
\end{aligned}
\tag{14}
$$

where A is the collection of events that the total degradation is less than H and all shock magnitudes are less than the corresponding failure thresholds. B_1 is the collection of events that the total degradation is less than L and $N(t)$ shocks occur before t; B_2 is the collection of events that the total degradation is no less than L, $N(t_*)$ shocks occur before t_*, and $N(t\text{-}t_*)$ shocks occur between t_* and t. A, B_1, and B_2 can be expressed as:

$$
A = \left(Xs(t) < H, \bigcap_{j=1}^{N(t_1)}\left(W_j<D_1\right), \bigcap_{j=N(t_1)+1}^{N(t)}\left(W_j<D_2\right)\right) \tag{15}
$$

$$
B_1 = \bigcup_{n=0}^{\infty}\left(Xs(t) < L, N(t)=n\right) \tag{16}
$$

$$
B_2 = \bigcup_{n=0}^{\infty}\bigcup_{n_1=0}^{n}\left(Xs(t) \ge L, N(t_*)=n_1, N(t-t_*)=n-n_1\right) \tag{17}
$$

If the degradation value of a component is less than L, then the probability that the component survives in the first case is:

$$
\begin{aligned}
P\left(AB_1\right) &= \sum_{n=0}^{\infty} P\left(Xs(t)<L, N(t)=n, \bigcap_{j=1}^{n}\left(W_j<D_1\right)\right) \\
&= \sum_{n=0}^{\infty} P\left(X(t)+S(t)<L \mid N(t)=n\right) P\left(N(t)=n\right) \prod_{j=1}^{n} P\left(W_j<D_1\right) \\
&= P\left(X(t)<L \mid N(t)=0\right) P\left(N(t)=0\right) \\
&\quad + \sum_{n=1}^{\infty} P\left(X(t)+S(t)<L \mid N(t)=n\right) P\left(N(t)=n\right) P\left(W_j<D_1\right)^n
\end{aligned}
\tag{18}
$$

If the degradation value of a component is no less than L, then the probability that the component survives in the second case is:

$$
\begin{aligned}
P\left(AB_2\right) &= \sum_{n=0}^{\infty}\sum_{n_1=0}^{n} P\left(L \le Xs\left(t\right) < H, N\left(t_*\right) = n_1, N\left(t - t_*\right) = n - n_1, \bigcap_{j=1}^{n_1}\left(W_j < D_1\right), \bigcap_{j=n_1+1}^{n}\left(W_j < D_2\right)\right) \\
&= P\left(L \le X\left(t\right) < H \middle| N\left(t=0\right)\right) P\left(N\left(t\right) = 0\right) \\
&+ \sum_{n=1}^{\infty}\sum_{n_1=0}^{n} \int_0^t P\left(L \le X\left(t\right) + S\left(t\right) < H \middle| N\left(t\right) = n\right) P\left(N\left(t_*\right) = n_1\right) \\
&\cdot P\left(N\left(t - t_*\right) = n - n_1\right) \prod_{j=1}^{n_1} P\left(W_j < D_1\right) \prod_{j=n_1+1}^{n} P\left(W_j < D_2\right) f\left(t_* \middle| N\left(t_*\right) = n_1\right) dt_*
\end{aligned}
\tag{19}
$$

where t_* is the time when the degradation value of components reaches L, $f(t_*|N(t_*) = n_1)$ is the conditional PDF of t_* when the number of shocks is n_1.

$$
f\left(t_* \middle| N\left(t_*\right) = n_1\right) = \frac{\partial F\left(t_* \middle| N\left(t_*\right) = n_1\right)}{\partial t_*}
\tag{20}
$$

where $F(t_*|N(t_*) = n_1)$ is the conditional CDF of t_*, expressed as:

$$
\begin{aligned}
F\left(t_* \middle| N\left(t_*\right) = n_1\right) &= P\left(T \le t_* \middle| N\left(t_*\right) = n_1\right) \\
&= P\left(Xs\left(t_*\right) \ge L \middle| N\left(t_*\right) = n_1\right) \\
&= 1 - P\left(Xs\left(t_*\right) \le L \middle| N\left(t_*\right) = n_1\right)
\end{aligned}
\tag{21}
$$

where

$$
\begin{aligned}
P\left(Xs\left(t_*\right) \le L \middle| N\left(t_*\right) = n_1\right) &= \begin{cases} P\left(X\left(t_*\right) + S\left(t_*\right) \le L \middle| N\left(t_*\right) = n_1, n_1 \ne 0\right) \\ P\left(X\left(t_*\right) \le L \middle| N\left(t_*\right) = n_1, n_1 = 0\right) \end{cases} \\
&= \begin{cases} \int_0^L \int_0^{L-x} g_{S(t)}\left(u \middle| N\left(t_*\right) = n_1, n_1 \ne 0\right) f_{X(t)}\left(x \middle| \mu, \lambda, p, t_*\right) du dx \\ \int_0^L f_{X(t)}\left(x \middle| \mu, \lambda, p, t_*\right) dx \end{cases}
\end{aligned}
$$

The reliability of a component experiencing dependent CFPs with changing failure thresholds can be derived as:

$$
\begin{aligned}
R_{\text{dep}}\left(t\right) &= P\left(AB_1\right) + P\left(AB_2\right) \\
&= \int_0^L f_{X(t)}\left(x \middle| \mu, \lambda, p, t\right) dx \cdot \exp\left(-\theta t\right) + \sum_{n=1}^{\infty} \int_0^L \int_0^{L-u} f_{X(t)}\left(x \middle| \mu, \lambda, p, t\right) g_{S(t)}\left(u \middle| N\left(t\right) = n\right) dx du \\
&\cdot r(t) r_1(t, n) r_2(W_j | D_1, n) + \int_L^H \left[f_{X(t)}\left(x \middle| \mu, \lambda, p, t\right) dx \right] \cdot r(t) \\
&+ \sum_{n=1}^{\infty}\sum_{n_1=0}^{n} \int_0^t \int_L^H \int_{L-u}^{H-u} \left[f_{X(t)}\left(x \middle| \mu, \lambda, p, t\right) g_{S(t)}\left(u \middle| N\left(t\right) = n\right) dx du \right] \\
&\cdot r(t_*) r_1(t_*, n_1) r(t - t_*) r_1(t - t_*, n - n_1) r_2(W_j | D_1, n_1) r_2(W_j | D_2, n - n_1) f\left(t_* \middle| N\left(t_*\right) = n_1\right) dt_* \\
&= \int_0^H \left[f_{X(t)}\left(x \middle| \mu, \lambda, p, t\right) dx \right] \cdot \exp\left(-\theta t\right) + \sum_{n=1}^{\infty} \int_0^L \int_0^{L-u} \left[f_{X(t)}\left(x \middle| \mu, \lambda, p, t\right) g_{S(t)}\left(u \middle| N\left(t\right) = n\right) dx du \right] \\
&\cdot r(t) r_1(t, n) r_2(W_j | D_1, n) + \sum_{n=1}^{\infty}\sum_{n_1=0}^{n} \int_0^t \int_L^H \int_{L-u}^{H-u} \left[f_{X(t)}\left(x \middle| \mu, \lambda, p, t\right) g_{S(t)}\left(u \middle| N\left(t\right) = n\right) dx du \right] \\
&\cdot r(t_*) r_1(t_*, n_1) r(t - t_*) r_1(t - t_*, n - n_1) r_2(W_j | D_1, n_1) r_2(W_j | D_2, n - n_1) f\left(t_* \middle| N\left(t_*\right) = n_1\right) dt_*
\end{aligned}
\tag{22}
$$

3. Reliability analysis for systems based on DP, CFPs, and survival signature

To complete the system reliability analysis, component reliability analysis and system structure analysis are two essential parts. In this section, the survival signature is utilized to complete the structure analysis, and new generalized reliability models are proposed for arbitrary systems with general structures experiencing multiple DPs and CFPs.

3.1 System reliability based on survival signature

Suppose that a system is composed of K types of components. Let m_k represent the number of components of type k, where $k = 1, 2, \ldots, K$. l_k denotes the number of functioning components of type k. Let vector $\boldsymbol{Y}$ represent the operational state of the components, which can be denoted as $\boldsymbol{Y} = (\boldsymbol{Y_1}, \boldsymbol{Y_2}, \ldots, \boldsymbol{Y_k}, \ldots, \boldsymbol{Y_K})$, where $\boldsymbol{Y_k}$ denotes the operational status of the components of type k, $\boldsymbol{Y_k} = \left(y_{1_k}, y_{2_k}, \cdots, y_{i_k}, \cdots, y_{m_k}\right)$, i_k means the ith component of type k. If the i_k-th component functions, then $y_{i_k} = 1$ and $y_{i_k} = 0$, if it does not function. The structure function can be expressed as:

$$\phi(\boldsymbol{Y}) = \begin{cases} 1 & \text{if the system works} \\ 0 & \text{if the system fails} \end{cases} \tag{23}$$

The survival signature of the system is:

$$\Phi_S(l_1, l_2, \cdots, l_K) = \left[\prod_{k=1}^{K} \binom{m_k}{l_k}\right]^{-1} \times \sum_{\boldsymbol{Y} \in \boldsymbol{S}_{l_1,\ldots,l_K}} \phi(\boldsymbol{Y}) \tag{24}$$

where $\boldsymbol{S}_{l_1,\ldots,l_K} = \left[\boldsymbol{S}_{l_1}, \ldots, \boldsymbol{S}_{l_k}, \ldots, \boldsymbol{S}_{l_K}\right]$, $\boldsymbol{S}_{l_1,\ldots,l_K}$ is the set of all possible state vectors of the system, $\boldsymbol{S}_{l_k}$ is the set of all possible state vectors of the components of type k when the number of working components of the k-th type is l_k.

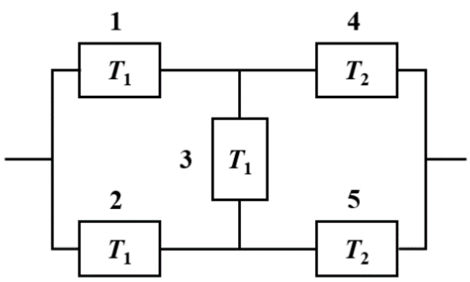

Fig. 5 A bridge system

To illustrate the computation of the survival signature, a bridge system consisting of two types of components is presented in **Fig. 5.** The survival signature of this system can be calculated by Eq. (24). For the system in **Fig. 5**, there are $(m_1+1) \times (m_2+1) = 4 \times 3 = 12$ combinations of l_1 and l_2, the survival signature is given in **Table 1**. An example for calculating the vector $\boldsymbol{Y}$ and the structure function $\phi(\boldsymbol{Y})$ is provided, and the results are shown in **Table 2**.

Table 1 Survival signature of the bridge system

l_1	l_2	$\Phi_s(l_1, l_2)$	l_1	l_2	$\Phi_s(l_1, l_2)$
0	0	0	2	1	1
1	0	0	3	1	1
2	0	0	0	2	0
3	0	0	1	2	2/3
0	1	0	2	2	1
1	1	1/3	3	2	1

Table 2 The vector $\boldsymbol{Y}$ and $\phi(\boldsymbol{Y})$ when $l_1 = 1$ and $l_2 = 2$

l_1	l_2	$\boldsymbol{Y}$	$\phi(\boldsymbol{Y})$	$\Phi_S(l_1, l_2)$
		(0,0,1,1,1)	0	
1	2	(0,1,0,1,1)	1	2/3
		(1,0,0,1,1)	1	

Assume the components of different types fail independently, then the reliability of the system can be expressed as:

$$R_{\text{syst}}(t) = P(T_{\text{syst}} > t) = \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left[\Phi_S(l_1, l_2, \cdots, l_K)\prod_{k=1}^{K}P(C_{t,k} = l_k)\right] \tag{25}$$

where $P(C_{t,k} = l_k)$ is the probability that l_k out of m_k components function at time t. If the failure times of components of the same type are independent and identically distributed, then

$$P(C_{t,k} = l_k) = \binom{m_k}{l_k}\left[R_k(t)\right]^{l_k}\left[1 - R_k(t)\right]^{m_k - l_k} \tag{26}$$

where $R_k(t)$ is the reliability of a component of type k.

The system reliability can be derived as:

$$R_{\text{syst}}(t) = \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left[\Phi_S(l_1, l_2, \cdots, l_K)\prod_{k=1}^{K}\binom{m_k}{l_k}\left[R_k(t)\right]^{l_k}\left[1 - R_k(t)\right]^{m_k - l_k}\right] \tag{27}$$

3.2 System reliability analysis based on DPs, CFPs and survival signature

Case 1: Systems experiencing pure DPs without CFPs

If the components of a system fail due to pure DPs without shocks, the probability that the system survives from soft failures can be derived as:

$$\begin{aligned} R_{\text{syst}}(t) &= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left\{\Phi_S(l_1,l_2,\cdots,l_K)\prod_{k=1}^{K}\binom{m_k}{l_k}\left[R_k(t)\right]^{l_k}\left[1-R_k(t)\right]^{m_k-l_k}\right\} \\ &= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left\{\Phi_S(l_1,l_2,\cdots,l_K)\prod_{k=1}^{K}\binom{m_k}{l_k}\left[R_{S\_k}(t)\right]^{l_k}\left[1-R_{S\_k}(t)\right]^{m_k-l_k}\right\} \\ &= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left\{\Phi_S(l_1,l_2,\cdots,l_K)\prod_{k=1}^{K}\binom{m_k}{l_k}\left[P\left(X_k(t)<H_k\right)\right]^{l_k}\left[1-P\left(X_k(t)<H_k\right)\right]^{m_k-l_k}\right\} \end{aligned} \tag{28}$$

where $R_{S\_k}(t)$ is the reliability of the components of type k, which are subject to pure DPs. $X_k(t)$ is the degradation value of the components of type k, and H_k is the soft failure threshold of the components of type k. If the degradation of components follows the Wiener, Gamma, or inverse Gaussian processes, which are the special cases of the TED process, then the reliability of a system experiencing pure DPs can be derived as:

$$\begin{aligned} R_{\text{syst}}(t) &= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left\{\Phi_S(l_1,l_2,\cdots,l_K)\prod_{k=1}^{K}\binom{m_k}{l_k}\left[P\left(X_k(t)<H_k\right)\right]^{l_k}\left[1-P\left(X_k(t)<H_k\right)\right]^{m_k-l_k}\right\} \\ &= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left\{\Phi_S(l_1,l_2,\cdots,l_K)\prod_{k=1}^{K}\binom{m_k}{l_k}\left[\int_0^{H_k} f_{X_k(t)}\left(x_k\,|\mu_k,\lambda_k,p_k,t\right)dx_k\right]^{l_k}\right. \\ &\quad \left.\cdot\left[1-\int_0^{H_k} f_{X_k(t)}\left(x_k\,|\mu_k,\lambda_k,p_k,t\right)dx_k\right]^{m_k-l_k}\right\} \end{aligned} \tag{29}$$

where $X_k(t) \sim TED(\mu_k t, \lambda_k, p_k)$, $p_k \in (-\infty, 0] \cup [1, +\infty)$, μ_k and λ_k respectively are the drift parameter and diffusion parameter of the components of type k, $f_{X_k(t)}(\cdot)$ is the PDF of the degradation value $X_k(t)$, which is:

$$f_{X_k(t)}\left(x_k\,|\mu_k,\lambda_k,p_k,t\right) = \int_0^{H_k}\sqrt{\frac{\lambda_k}{2\pi t^{1-p_k} x_k^{p_k}}}\exp\left\{-\frac{\lambda_k t}{2}d\left(\frac{x_k}{t};\mu_k\right)\right\}dx_k \tag{30}$$

where,

$$d\left(\frac{x_k}{t};\mu_k\right) = \begin{cases} 2\left[\dfrac{x_k}{t}\ln\left(\dfrac{x_k}{\mu_k t}\right)-\dfrac{x_k}{t}+\mu_k\right], p_k = 1 \\ 2\left[\ln\left(\dfrac{\mu_k t}{x_k}\right)+\dfrac{x_k}{\mu_k t}-1\right], p_k = 2 \\ 2\left[\dfrac{x_k^{2-p_k}}{t^{2-p_k}(1-p_k)(2-p_k)}-\dfrac{x_k\mu_k^{(1-p_k)}}{t(1-p_k)}+\dfrac{\mu_k^{(2-p_k)}}{(2-p_k)}\right], p_k \neq 1,2 \end{cases}$$

Case 2: Systems experiencing independent CFPs

For some systems, the components may experience independent degradation-shock processes. For example, a concrete column of a bridge may fail due to a soft failure caused by corrosion or a hard failure caused by sudden traffic crashes or earthquakes. The probability that a system survives independent CFPs is:

$$
\begin{aligned}
R_{\text{syst}}(t) &= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left\{\Phi_{\text{S}}(l_1,l_2,\cdots,l_K)\prod_{k=1}^{K}\binom{m_k}{l_k}\left[R_k(t)\right]^{l_k}\left[1-R_k(t)\right]^{m_k-l_k}\right\} \\
&= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left\{\Phi_{\text{S}}(l_1,l_2,\cdots,l_K)\prod_{k=1}^{K}\binom{m_k}{l_k}\left[R_{\text{ind}\_k}(t)\right]^{l_k}\left[1-R_{\text{ind}\_k}(t)\right]^{m_k-l_k}\right\} \\
&= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left\{\Phi_{\text{S}}(l_1,l_2,\cdots,l_K)\prod_{k=1}^{K}\binom{m_k}{l_k}\left[\sum_{n_k=0}^{\infty}P\left(X_k(t)<H_k, N_k(t)=n_k, \bigcap_{j_k=1}^{n_k}\left(W_{j_k}<D_k\right)\right)\right]^{l_k}\right. \\
&\quad \left.\cdot\left[1-\sum_{n_k=0}^{\infty}P\left(X_k(t)<H_k, N_k(t)=n_k, \bigcap_{j_k=1}^{n_k}\left(W_{j_k}<D_k\right)\right)\right]^{m_k-l_k}\right\} \\
&= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left\{\Phi_{\text{S}}(l_1,l_2,\cdots,l_K)\prod_{k=1}^{K}\binom{m_k}{l_k}\left[\sum_{n_k=0}^{\infty}P\left(X_k(t)<H_k\right)P\left(N_k(t)=n_k\right)P\left(W_{j_k}<D_k\right)^{n_k}\right]^{l_k}\right. \\
&\quad \left.\left[1-\sum_{n_k=0}^{\infty}P\left(X_k(t)<H_k\right)P\left(N_k(t)=n_k\right)P\left(W_{j_k}<D_k\right)^{n_k}\right]^{m_k-l_k}\right\}
\end{aligned}
\tag{31}
$$

where $R_{\text{ind}\_k}(t)$ is the reliability of the components of type k, which are subject to independent CFPs, $N_k(t)$ is the number of shocks on the components of type k during $(0, t]$, W_{j_k} is the magnitude of the jth shock on the components of type k, D_k is the hard failure threshold of the component of type k. If the components are exposed to the independent generalized TED process and the extreme shock process, then the system reliability can be derived as:

$$
\begin{aligned}
R_{\text{syst}}(t) &= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left\{\Phi_{\text{S}}(l_1,l_2,\cdots,l_K)\prod_{k=1}^{K}\binom{m_k}{l_k}\right. \\
&\quad \cdot\left[\sum_{n_k=0}^{\infty}P\left(X_k(t)<H_k\right)P\left(N_k(t)=n_k\right)P\left(W_{j_k}<D_k\right)^{n_k}\right]^{l_k} \\
&\quad \left.\cdot\left[1-\sum_{n_k=0}^{\infty}P\left(X_k(t)<H_k\right)P\left(N_k(t)=n_k\right)P\left(W_{j_k}<D_k\right)^{n_k}\right]^{m_k-l_k}\right\} \\
&= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left\{\Phi_{\text{S}}(l_1,l_2,\cdots,l_K)\prod_{k=1}^{K}\binom{m_k}{l_k}\right. \\
&\quad \cdot\left[\sum_{n_k=0}^{\infty}\int_0^{H_k}\left[f_{X_k(t)}\left(x_k\left|\mu_k,\lambda_k,p_k,t\right.\right)dx_k\right]\cdot r_k(t)r_{1k}(t,n_k)r_{2k}(W_{j_k}\left|D_k,n_k\right.)\right]^{l_k} \\
&\quad \left.\cdot\left[1-\sum_{n_k=0}^{\infty}\int_0^{H_k}\left[f_{X_k(t)}\left(x_k\left|\mu_k,\lambda_k,p_k,t\right.\right)dx_k\right]\cdot r_k(t)r_{1k}(t,n_k)r_{2k}(W_{j_k}\left|D_k,n_k\right.)\right]^{m_k-l_k}\right\}
\end{aligned}
\tag{32}
$$

where μ_{wk} and σ_{wk} are the mean and variance of the magnitudes of the shocks on the components of type k, θ_k is the intensity of the shock on the components of type k, and $r_k(t)=\exp(-\theta_k t)$, $r_{1k}(t,n_k)=\dfrac{(\theta_k t)^{n_k}}{n_k!}$,

$$
r_{2k}\left(W_{j_k}\left|D_k,n_k\right.\right)=\Phi\left(\frac{D_k-\mu_{Wk}}{\sigma_{Wk}}\right)^{n_k}
$$

Case 3: Systems experiencing dependent CFPs

The components of some systems may operate under dependent degradation-shock processes. For example, as the crack size of gear wheel teeth gets larger, the gear wheel teeth become more susceptible to fatigue fracture when experiencing random shocks. If the components suffer from the dependent CFPs, in which the random shocks can cause sudden increments to the degradation, and increasing degradation levels can reduce the capability of the components to resist random shocks, then the system reliability calculated based on the survival signature can be derived as:

$$\begin{aligned} R_{\text{syst}}(t) &= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left\{\Phi_{\text{S}}(l_1,l_2,\cdots,l_K)\prod_{k=1}^{K}\binom{m_k}{l_k}\left[R_k(t)\right]^{l_k}\left[1-R_k(t)\right]^{m_k-l_k}\right\} \\ &= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left\{\Phi_{\text{S}}(l_1,l_2,\cdots,l_K)\prod_{k=1}^{K}\binom{m_k}{l_k}\left[R_{\text{dep}\_k}(t)\right]^{l_k}\left[1-R_{\text{dep}\_k}(t)\right]^{m_k-l_k}\right\} \\ &= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left\{\Phi_{\text{S}}(l_1,l_2,\cdots,l_K)\prod_{k=1}^{K}\binom{m_k}{l_k}\left[P(A_kB_{1k})+P(A_kB_{2k})\right]^{l_k}\right. \\ &\quad \left.\cdot\left[1-P(A_kB_{1k})-P(A_kB_{2k})\right]^{m_k-l_k}\right\} \end{aligned} \tag{33}$$

where A_kB_{1k} and A_kB_{2k} are two independent and mutually exclusive events of the components of type k, which are exposed to dependent CPFs. The events A_k, B_{1k}, and B_{2k} can be expressed as:

$$A_k = \left(Xs_k(t) < H_k, \bigcap_{j_k=1}^{N_k(t_{*k})}\left(W_{j_k} < D_{1k}\right), \bigcap_{j_k=N_k(t_{*k})+1}^{N_k(t)}\left(W_{j_k} < D_{2k}\right)\right) \tag{34}$$

$$B_{1k} = \bigcup_{n_k=0}^{\infty}\left(Xs_k(t) < L_k, N_k(t) = n_k\right) \tag{35}$$

$$B_{2k} = \bigcup_{n_k=0}^{\infty}\bigcup_{n_{1k}=0}^{n_k}\left(Xs_k(t) \geq L_k, N_k(t_{*k}) = n_{1k}, N_k(t-t_{*k}) = n_k - n_{1k}\right) \tag{36}$$

where L_k is the degradation level of the components of type k, at which the hard failure threshold changes from D_{1k} to D_{2k}. t_{*k} is the time when the degradation level of the components of type k reaches L_k. n_k and n_{1k} are respectively the number of shocks on the components of type k during (0, t_{*k}] and (t_{*k}, t]. The calculation procedures of P(A_kB_{1k}) and P(A_kB_{2k}) are similar to Eqs. (18-19), and the details are provided in the appendix A. If the components of type k are subject to the generalized TED process and the extreme shock process, then the system reliability is:

$$
\begin{aligned}
R_{\text{syst}}(t) &= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left\{\Phi_{\text{S}}(l_1,l_2,\cdots,l_K)\prod_{k=1}^{K}\binom{m_k}{l_k}\left[P\left(A_kB_{1k}\right)+P\left(A_kB_{2k}\right)\right]^{l_k}\right.\\
&\left.\cdot\left[1-P\left(A_kB_{1k}\right)-P\left(A_kB_{2k}\right)\right]^{m_k-l_k}\right\}\\
&= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left\{\Phi_{\text{S}}(l_1,l_2,\cdots,l_K)\prod_{k=1}^{K}\binom{m_k}{l_k}\left[\int_0^{H_k}\left[f_{X_k(t)}\left(x_k\left|\mu_k,\lambda_k,p_k,t\right.\right)dx_k\right]\cdot\exp\left(-\theta_k t\right)\right.\right.\\
&+\sum_{n_k=1}^{\infty}\int_0^{L_k}\int_0^{L_k-u_k}\left[f_{X_k(t)}\left(x_k\left|\mu_k,\lambda_k,p_k,t\right.\right)g_{S_k(t)}\left(u_k\left|N_k(t)=n_k\right.\right)dx_k du_k\right]\\
&\cdot r_k(t)r_{1k}(t,n_k)r_{2k}(W_{j_k}\left|D_{k1},n_k\right.)+\sum_{n_k=1}^{\infty}\sum_{n_{k1}=0}^{n_k}\int_0^{t}\int_{L_k}^{H_k}\int_{L_k-u_k}^{H_k-u_k}\left[f_{X_k(t)}\left(x_k\left|\mu_k,\lambda_k,p_k,t\right.\right)\right.\\
&\left.\cdot g_{S_k(t)}\left(u_k\left|N_k(t)=n_k\right.\right)dx_k du_k\right]\cdot r_k(t_*)r_{1k}(t_*,n_{1k})r_k(t-t_*)r_{1k}(t-t_*,n_k-n_{1k})\\
&\left.\cdot r_{2k}(W_{j_k}\left|D_{1k},n_{1k}\right.)r_{2k}(W_{j_k}\left|D_{2k},n_k-n_{1k}\right.)f\left(t_{*k}\left|N_k\left(t_{*k}\right)=n_{1k}\right.\right)dt_{*k}\right]^{l_k}\\
&\cdot\left[1-\int_0^{H_k}\left[f_{X_k(t)}\left(x_k\left|\mu_k,\lambda_k,p_k,t\right.\right)dx_k\right]\cdot r_k(t)-\right.\\
&\sum_{n_k=1}^{\infty}\int_0^{L_k}\int_0^{L_k-u_k}\left[f_{X_k(t)}\left(x_k\left|\mu_k,\lambda_k,p_k,t\right.\right)g_{S_k(t)}\left(u_k\left|N_k(t)=n_k\right.\right)dx_k du_k\right]\\
&\cdot r_k(t)r_{1k}(t,n_k)r_{2k}(W_{j_k}\left|D_{k1},n_k\right.)-\sum_{n_k=1}^{\infty}\sum_{n_{k1}=0}^{n_k}\int_0^{t}\int_{L_k}^{H_k}\int_{L_k-u_k}^{H_k-u_k}\left[f_{X_k(t)}\left(x_k\left|\mu_k,\lambda_k,p_k,t\right.\right)\right.\\
&\left.\cdot g_{S_k(t)}\left(u_k\left|N_k(t)=n_k\right.\right)dx_k du_k\right]r_k(t_*)r_{1k}(t_*,n_{1k})r_k(t-t_*)r_{1k}(t-t_*,n_k-n_{1k})\\
&\left.\left.\cdot r_{2k}(W_{j_k}\left|D_{1k},n_{1k}\right.)r_{2k}(W_{j_k}\left|D_{2k},n_k-n_{1k}\right.)f\left(t_{*k}\left|N_k\left(t_{*k}\right)=n_{1k}\right.\right)dt_{*k}\right]^{m_k-l_k}\right\}
\end{aligned}
\tag{37}
$$

where $g_{S_k(t)}\left(u_k\left|N_k(t)=n_k,n_k\neq 0\right.\right)=\frac{1}{\sqrt{2\pi n_k\sigma_{Zk}^2}}\exp\left[-\frac{\left(u_k-n_k\mu_{Zk}\right)^2}{2n_k\sigma_{Zk}^2}\right]$, which is the PDF of the total shock damage of the components of type k and $\mu_{Zk}=a_k\mu_{Wk}$, $\sigma_{Zk}=a_k\sigma_{Wk}$, a_k is a constant representing the relation between the damages and magnitudes of the shocks on the components of type k.

Case 4: Systems experiencing multiple DPs and CFPs

In the cases above, the generalized reliability models for systems suffering from different degradation-shock processes are presented. But different types of components in one system are considered to be subject to similar degradation and shock processes. However, for some practical systems, different types of components may operate under different working conditions and experience different degradation and shock processes, such as the automotive braking system, which is illustrated in detail in Section 4. The generalized reliability model of a system subject to multiple degradation and shock processes can be expressed as:

$$
\begin{aligned}
R_{\text{syst}}(t) &= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left\{\Phi_{\text{S}}(l_1,l_2,\cdots,l_K)\prod_{k=1}^{K}\binom{m_k}{l_k}\left[R_k(t)\right]^{l_k}\left[1-R_k(t)\right]^{m_k-l_k}\right\} \\
&= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left\{\Phi_{\text{S}}(l_1,l_2,\cdots,l_K)\prod_{k=1}^{k_1}\binom{m_k}{l_k}\left[R_{S\_k}(t)\right]^{l_k}\left[1-R_{S\_k}(t)\right]^{m_k-l_k}\right. \\
&\quad \left.\cdot\prod_{k=k_1+1}^{k_2}\binom{m_k}{l_k}\left[R_{\text{ind}\_k}(t)\right]^{l_k}\left[1-R_{\text{ind}\_k}(t)\right]^{m_k-l_k}\prod_{k=k_2+1}^{K}\binom{m_k}{l_k}\left[R_{\text{dep}\_k}(t)\right]^{l_k}\left[1-R_{\text{dep}\_k}(t)\right]^{m_k-l_k}\right\} \\
&= \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\left\{\Phi_{\text{S}}(l_1,l_2,\cdots,l_K)\prod_{k=1}^{k_1}\binom{m_k}{l_k}\left[P\left(X_k(t)<H_k\right)\right]^{l_k}\left[1-P\left(X_k(t)<H_k\right)\right]^{m_k-l_k}\right. \\
&\quad \cdot\prod_{k=k_1+1}^{k_2}\binom{m_k}{l_k}\left[\sum_{n_k=0}^{\infty}P\left(X_k(t)<H_k,N_k(t)=n_k,\bigcap\nolimits_{j_k=1}^{n_k}\left(W_{j_k}<D_k\right)\right)\right]^{l_k} \\
&\quad \cdot\left[1-\sum_{n_k=0}^{\infty}P\left(X_k(t)<H_k,N_k(t)=n_k,\bigcap\nolimits_{j_k=1}^{n_k}\left(W_{j_k}<D_k\right)\right)\right]^{m_k-l_k} \\
&\quad \left.\cdot\prod_{k=k_2+1}^{K}\binom{m_k}{l_k}\left[P\left(A_kB_{1k}\right)+P\left(A_kB_{2k}\right)\right]^{l_k}\left[1-P\left(A_kB_{1k}\right)-P\left(A_kB_{2k}\right)\right]^{m_k-l_k}\right\}
\end{aligned}
\tag{38}
$$

where k_1 is number of types of components experiencing pure degradation without CFPs, k_2-k_1 is the number of types of components subject to independent CFPs, and K-k_2 is the number of types of components experiencing dependent CFPs. For example, if a system is composed of 3 types of components, and they are subject to pure degradation, independent and dependent CFPs respectively, then $k_1 = 1$, k_2-$k_1 = 1$, K-$k_2 = 1$, $k_2 = 2$, $K = 3$. If the component degradation follows the Wiener, Gamma, inverse Gaussian and other stochastic processes which are special cases of the generalized TED process, and the random shock process can be described as the extreme shock process, then the proposed system reliability in Eq. (38) can be expressed as:

$$
\begin{aligned}
R_{\mathrm{syst}}(t) = {} & \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2}\cdots\sum_{l_K=0}^{m_K}\bigg\{\Phi_{\mathrm{S}}(l_1,l_2,\cdots,l_K)\prod_{k=1}^{k_1}\binom{m_k}{l_k}\bigg[\int_0^{H_k}\Big[f_{X_k(t)}\big(x_k\,|\mu_k,\lambda_k,p_k,t\big)dx_k\Big]\bigg]^{l_k} \\
& \cdot\bigg[1-\int_0^{H_k}\Big[f_{X_k(t)}\big(x_k\,|\mu_k,\lambda_k,p_k,t\big)dx_k\Big]\bigg]^{m_k-l_k} \\
& \cdot\prod_{k=k_1}^{k_2}\binom{m_k}{l_k}\bigg[\sum_{n_k=0}^{\infty}\int_0^{H_k}\Big[f_{X_k(t)}\big(x_k\,|\mu_k,\lambda_k,p_k,t\big)dx_k\Big]\cdot r_k(t)r_{1k}(t,n_k)r_{2k}(W_{j_k}\,|D_k,n_k)\bigg]^{l_k} \\
& \cdot\bigg[1-\sum_{n_k=0}^{\infty}\int_0^{H_k}\Big[f_{X_k(t)}\big(x_k\,|\mu_k,\lambda_k,p_k,t\big)dx_k\Big]\cdot r_k(t)r_{1k}(t,n_k)r_{2k}(W_{j_k}\,|D_k,n_k)\bigg]^{m_k-l_k} \\
& \cdot\prod_{k=k_2+1}^{K}\binom{m_k}{l_k}\bigg[\int_0^{H_k}\Big[f_{X_k(t)}\big(x_k\,|\mu_k,\lambda_k,p_k,t\big)dx_k\Big]\cdot\exp(-\theta_k t) \\
& +\sum_{n_k=1}^{\infty}\int_0^{L_k}\int_0^{L_k-u_k}\Big[f_{X_k(t)}\big(x_k\,|\mu_k,\lambda_k,p_k,t\big)g_{S_k(t)}\big(u_k\,|N_k(t)=n_k\big)dx_k du_k\Big] \\
& \cdot r_k(t)r_{1k}(t,n_k)r_{2k}(W_{j_k}\,|D_{k1},n_k)+\sum_{n_k=1}^{\infty}\sum_{n_{k1}=0}^{n_k}\int_0^{t}\int_{L_k}^{H_k}\int_{L_k-u_k}^{H_k-u_k}\Big[f_{X_k(t)}\big(x_k\,|\mu_k,\lambda_k,p_k,t\big) \\
& \cdot g_{S_k(t)}\big(u_k\,|N_k(t)=n_k\big)dx_k du_k\Big]\cdot r_k(t_*)r_{1k}(t_*,n_{1k})r_k(t-t_*)r_{1k}(t-t_*,n_k-n_{1k}) \\
& \cdot r_{2k}(W_{j_k}\,|D_{1k},n_{1k})r_{2k}(W_{j_k}\,|D_{2k},n_k-n_{1k})f\big(t_{*k}\,|N_k(t_{*k})=n_{1k}\big)dt_{*k}\bigg]^{l_k} \\
& \bigg[1-\int_0^{H_k}\Big[f_{X_k(t)}\big(x_k\,|\mu_k,\lambda_k,p_k,t\big)dx_k\Big]\cdot r_k(t) \\
& -\sum_{n_k=1}^{\infty}\int_0^{L_k}\int_0^{L_k-u_k}\Big[f_{X_k(t)}\big(x_k\,|\mu_k,\lambda_k,p_k,t\big)g_{S_k(t)}\big(u_k\,|N_k(t)=n_k\big)dx_k du_k\Big] \\
& \cdot r_k(t)r_{1k}(t,n_k)r_{2k}(W_{j_k}\,|D_{k1},n_k)-\sum_{n_k=1}^{\infty}\sum_{n_{k1}=0}^{n_k}\int_{L_k}^{H_k}\int_{L_k-u_k}^{H_k-u_k}\Big[f_{X_k(t)}\big(x_k\,|\mu_k,\lambda_k,p_k,t\big) \\
& \cdot g_{S_k(t)}\big(u_k\,|N_k(t)=n_k\big)dx_k du_k\Big]r_k(t_*)r_{1k}(t_*,n_{1k})r_k(t-t_*)r_{1k}(t-t_*,n_k-n_{1k}) \\
& \cdot r_{2k}(W_{j_k}\,|D_{1k},n_{1k})r_{2k}(W_{j_k}\,|D_{2k},n_k-n_{1k})f\big(t_{*k}\,|N_k(t_{*k})=n_{1k}\big)dt_{*k}\bigg]^{m_k-l_k}\bigg\}
\end{aligned}
\tag{39}
$$

4. Numerical example

In this section, the proposed generalized reliability model for systems with multiple pure DPs and CFPs is illustrated by an automotive braking system, as shown in **Fig. 6**. The braking system consists of multiple types of components, among which the master cylinder, brake cylinders, brake pads, and handbrake cable are four critical types of components for braking. The simplified reliability block diagram is shown in **Fig. 7**, which is adapted from the work of Tavangar and Hashemi [32]. In **Fig. 7**, the components of types 1 and 2 are the master cylinder and brake cylinders. The cylinders fail mainly due to the wear of pistons and cylinders, and the wear is a pure DP without CFPs. The components of type 3 are brake pads. The brake pads are subject to independent CFPs, and the failure of brake pads is basically caused by wear or sudden reduction of friction coefficient due to thermal shocks. The components of type 4 represent the handbrake cable, which fails because of cumulative plastic deformation or snapping under sudden loads. The loads can cause abrupt increments in plastic deformation, and the increasing plastic deformation makes the cables more vulnerable to sudden loads.

The degradation-shock processes of the handbrake cable are dependent CFPs.

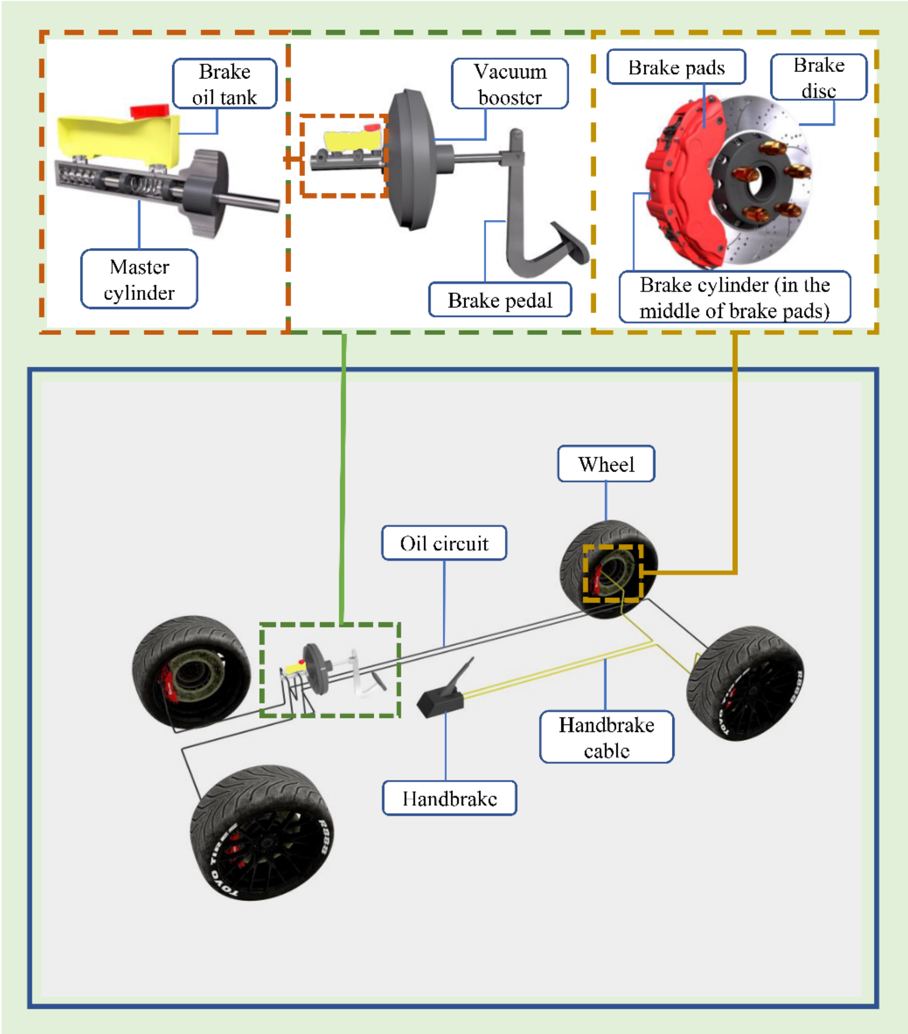

Fig. 6 An automotive braking system

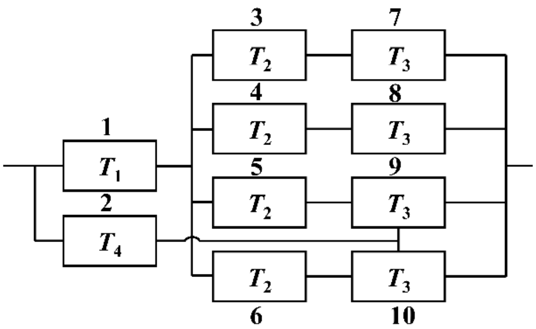

Fig. 7 Reliability block diagram of the braking system (Adapted from reference [32])

To evaluate the reliability of the automotive braking system in **Fig. 7**, the reliability of the composed four types of components needs to be calculated. The parameters of the degradation and shock processes of the components are assumed in **Table 3**. As shown in **Fig. 8**, the reliability of the first two types of components, failing because of pure degradation, can be calculated by substituting the related parameters in **Table 3** into Eq. (2), and the results are shown as the solid green and black lines. The reliability of brake pads, failing due to independent CFPs, can be calculated by substituting the parameters when $k = 3$ in **Table 3** into Eq. (7), and the result is shown as the solid purple line in **Fig. 8**. The reliability of the handbrake cable, failing due to dependent CFPs, can be calculated by substituting the parameters in **Table 3** in Eq. (22), and the result is shown as the solid blue line in **Fig. 8**.

Table 3 Parameters of the degradation and shock processes of the components

	Component type			
Parameters	$k = 1$	$k = 2$	$k = 3$	$k = 4$
H_k	5	6	9	7
L_k	-	-	-	4
p_k	2	2	3	3
λ_k	5	6	10	8
μ_k	1	1	2	2
D_{1k}	-	-	202	1
D_{2k}	-	-	-	0.8
μ_{wk}	-	-	200	1
σ_{wk}^2	-	-	10	0.01
a_k	-	-	-	1
θ_k			0.5	0.1

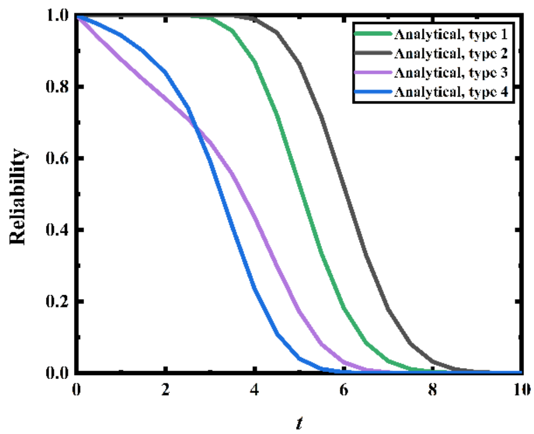

Fig. 8 Reliability of components calculated by the proposed method

To check the correctness of the derived equations of the proposed method, the theoretical reliability of multiple components and the PDF of t_* with a different number of shocks are compared with the MC simulation results. The flow charts of the MC simulation for the reliability calculation based on independent and dependent CFPs are shown in **Figs. 9-11**, respectively. The similar flow charts of the MC simulation for reliability calculation based on pure degradation processes are omitted. These can be obtained by analogy to the procedures shown in **Figs. 9-11**. The comparison results are shown with good agreements in **Figs. 12** and **13**, which indicates the potential application of the proposed reliability methods for components experiencing multiple DPs and CFPs.

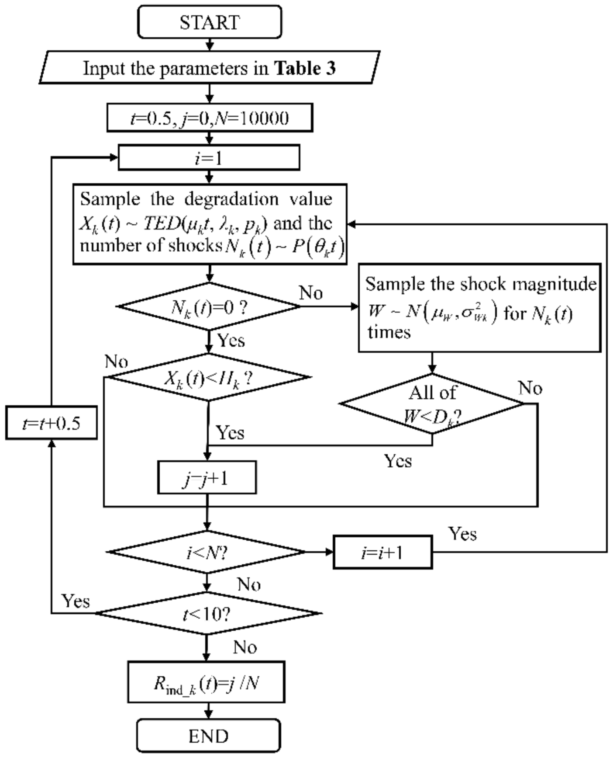

Fig. 9 Flow chart of the MC simulation for component reliability analysis based on independent CFPs

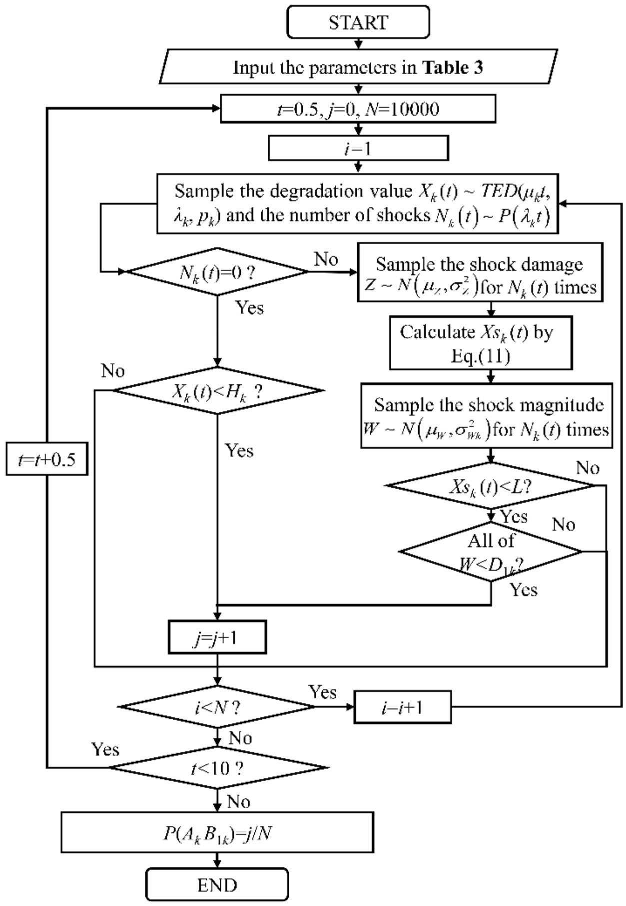

Fig. 10 Flow chart of the MC simulation for $P(A_k B_{1k})$ based on dependent CFPs

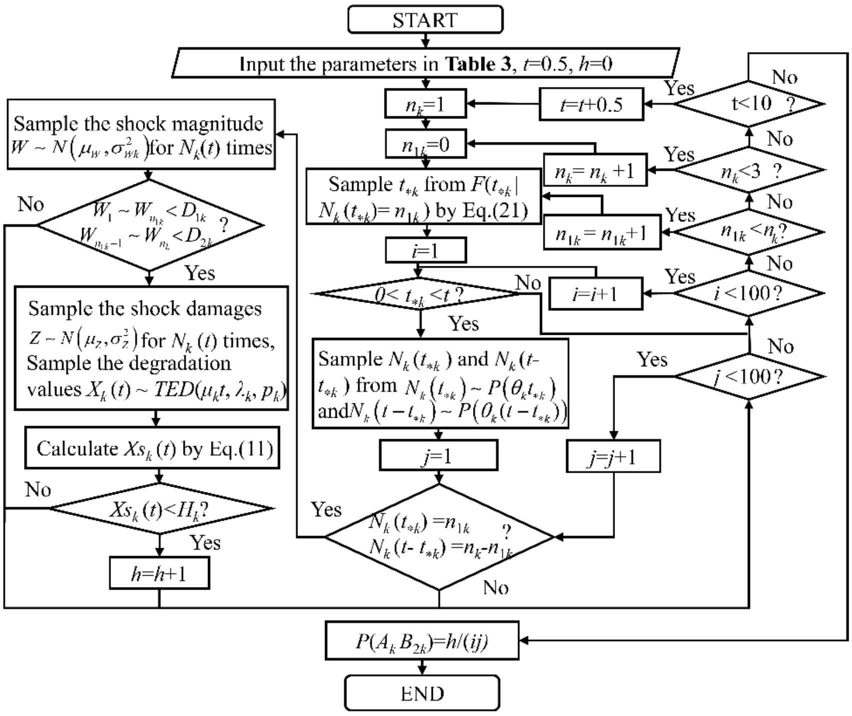

Fig. 11 MC simulation for $P(A_kB_{2k})$ based on dependent CFPs

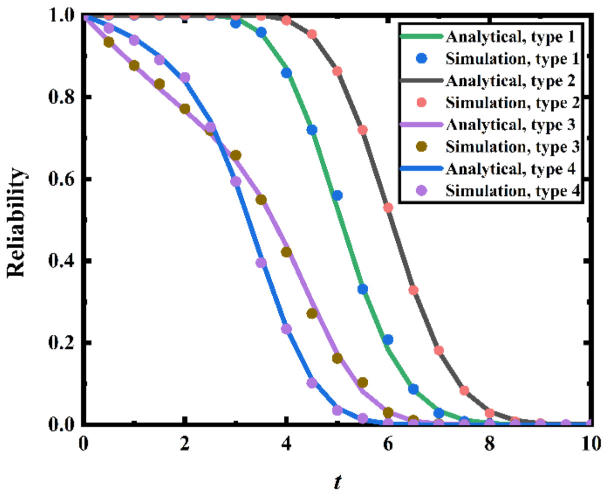

Fig. 12 Reliability of components calculated by the proposed method and MC simulation

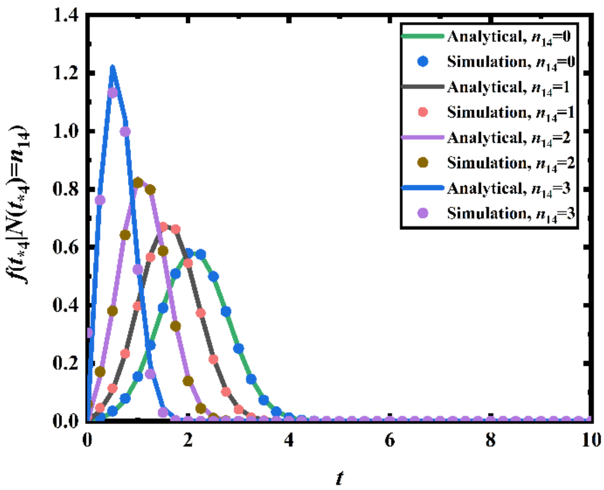

Fig. 13 The conditional PDF of t_* of the components of type 4 calculated by the proposed method and MC simulation

Before evaluating the system reliability, the system structure needs to be analyzed by the survival signature. The survival signature of the system is calculated by Eq. (24) and partly shown in **Table 3**. For the automotive braking system, the number of items of survival signature is $(m_1+1) \times (m_2+1) \times (m_3+1) \times (m_4+1) = 2\times5\times5\times2 = 100$. The complete survival signature of the system is available from the first author. Substituting the parameters in **Tables 3** and **4** into Eqs. (38) and (39), the reliability of the automotive braking system estimated by the proposed method, by considering the multiple types of components working under different DPs and CFPs, is shown as the solid blue line in **Fig. 14** and the reliabilities of multiple types of components are shown as the dashed lines in **Fig. 14**. Substituting the theoretical and MC simulation reliability of the components, shown in **Fig. 12**, into Eq. (38), the comparison results of the system reliability calculated by the proposed method and MC simulation method are shown in **Fig. 14**, showing that the theoretical reliability calculated by the proposed method is in good agreement with that obtained by MC simulation.

Table 4 The survival signature of the automotive braking system

l_1	l_2	l_3	l_4	$\Phi_s(l_1, l_2, l_3, l_4)$	l_1	l_2	l_3	l_4	$\Phi_s(l_1, l_2, l_3, l_4)$
0	0	1	1	0.5	1	3	3	1	1
0	0	2	1	0.83	1	3	4	1	1
0	0	3	1	1	1	4	1	1	1
0	0	4	1	1	1	4	2	1	1
0	1	1	1	0.5	1	4	3	1	1
⋮	⋮	⋮	⋮	⋮	1	4	4	1	1

Note: the complete table of the survival signature is available from the first author

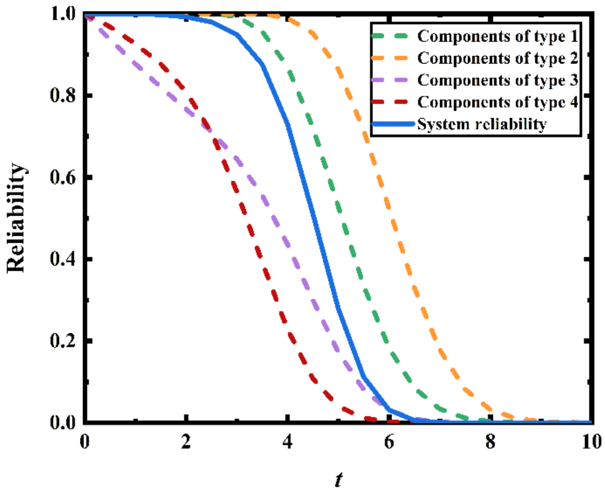

Fig. 14 Reliability of the braking system and components

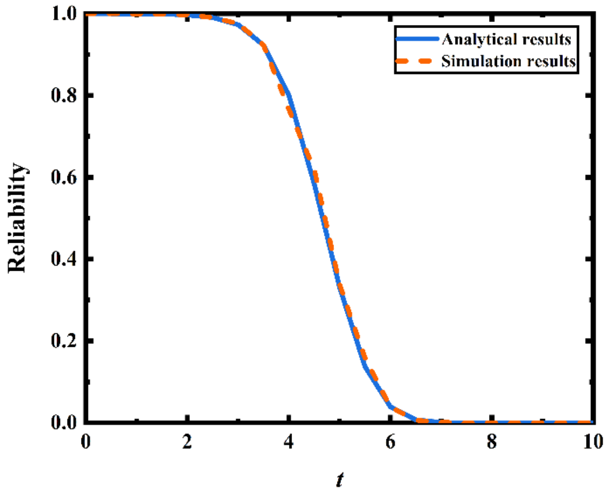

Fig. 15 System reliability calculated by the proposed method and MC simulation

As shown in **Figs. 16** and **17**, the reliabilities of components and the braking system are evaluated under different degradation-shock processes, including the pure DP without CFPs, independent and dependent CFPs. The differences among the reliability curves indicate that it is necessary to consider the practical dependence between the degradation and shock processes, instead of simply considering the degradation and shock processes of multiple components in one system being similar. For example, the reliability of the components of type 4, the handbrake cable, is affected by cumulative plastic deformation and random shocks caused by sudden loads. The increasing plastic deformation can cause the handbrake cable to snap more easily when suffering from sudden loads, and the sudden loads can cause abrupt increments in plastic deformation. But if the mutual effects between the degradation and shock processes are neglected, the reliability of the components of type 4 and the braking system is overestimated, shown as the green and red curves in **Figs. 16** and **17**. In contrast, for the components failing mainly due to pure degradation or independent competing failures, such as the brake cylinders and brake pads, the reliability can be underestimated if the components of the system are considered to work under similar dependent CFPs. Therefore, it is better to estimate the system reliability by considering the DPs and CFPs of multiple types of components according to their practical operational conditions and failure mechanisms.

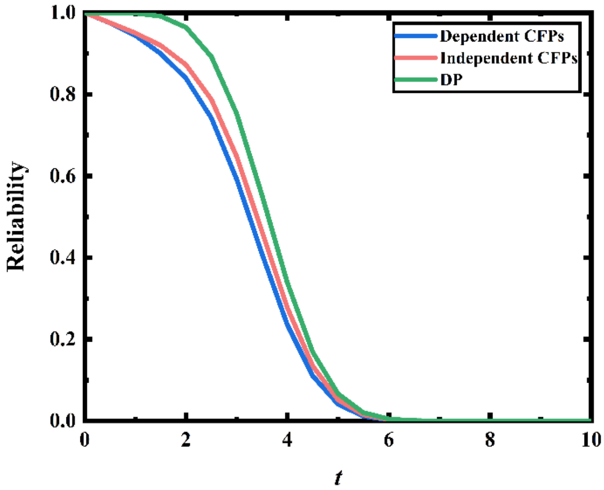

Fig. 16 Reliability of the components of type 4 under different failure modes

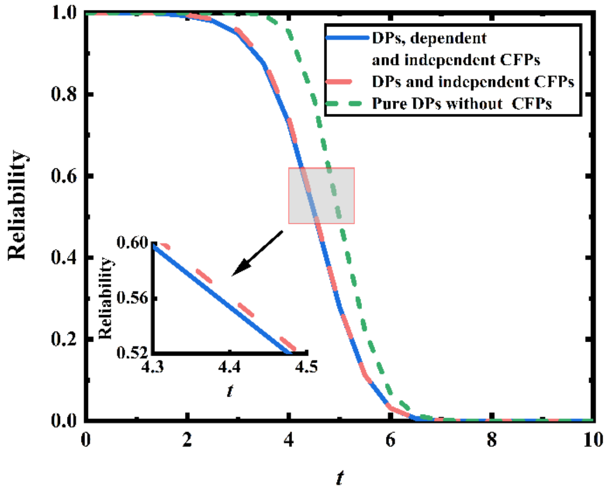

Fig. 17 Comparison of system reliability under different failure modes

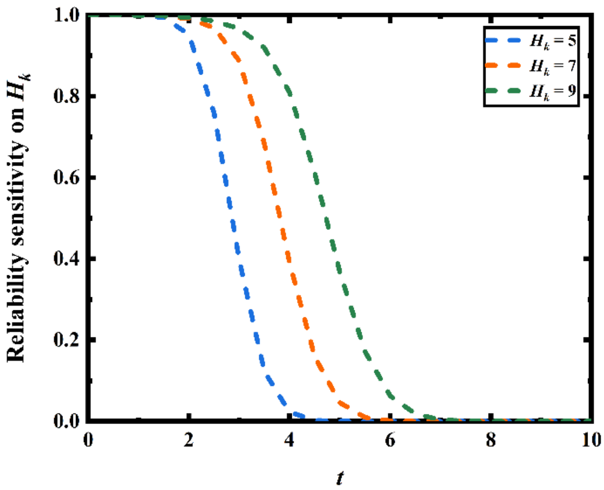

Fig. 18 Sensitivity of system reliability on H_k, where $k = 1, 2, 3, 4$

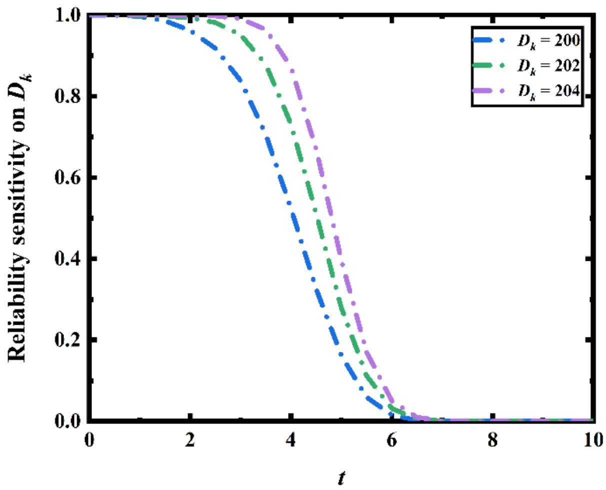

Fig. 19 Sensitivity of system reliability on D_k, where $k = 4$

Figs. 18-20 show the sensitivity of the system reliability on the parameters of DPs and CFPs. The reliabilities are calculated by substituting the parameters in **Table 3** into Eqs. (38) and (39), and the changed parameter values are shown in the legends in **Figs. 18-20**. As shown in **Fig. 18**, the system reliability shifts to the right with higher soft failure thresholds H_k, which can be explained by the fact that the larger the degradation of the components is allowed, the higher the reliability of the system is. For example, the thickness of the brake pads is about 15mm, and the brake pads need to be replaced if the wear of the brake pads reaches about 10mm. If the wear value H_k of the brake pads is allowed to be larger, then the reliability of the brake pads and the braking system is higher. The sensitivity of the reliability on the hard failure threshold D_k is shown in **Fig. 19**, the reliability shifts to the right with higher failure thresholds. Because if the thermostability of the brake pads is higher, then the reduction of the friction coefficient of brake pads is less when suffering from thermal shocks.

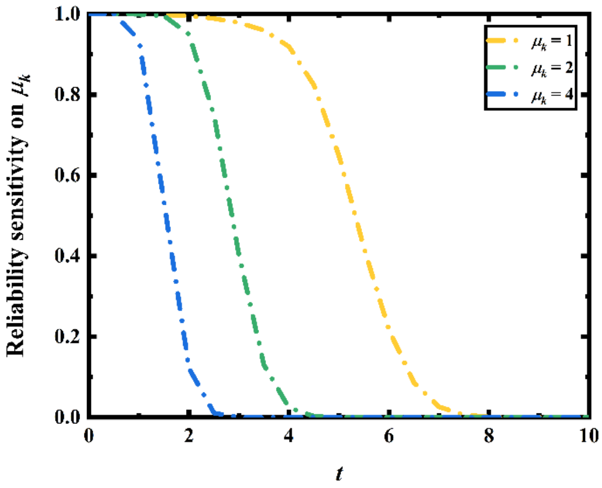

Fig. 20 Sensitivity of system reliability on μ_k, where $k = 1, 2, 3, 4$

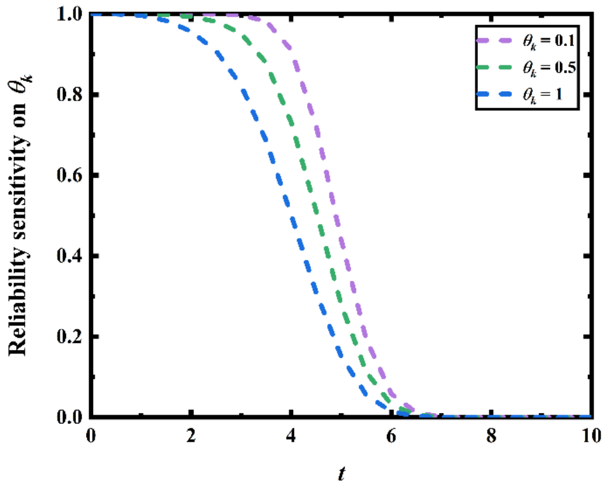

Fig. 21 Sensitivity of system reliability on θ_k, where $k = 3$ and 4

As shown in **Fig. 20**, the reliability of the system shifts to the left with increasing drift parameter μ, which represents the degradation speed of the components. This is because the faster the components degrade, the faster the system reliability declines. For example, if the wear resistance of the piston is better, then the brake oil is less likely to leak, and the braking system works with higher reliability. The sensitivity of the system reliability with regard to the shock intensity θ is shown in **Fig. 21**. The system reliability shifts to the left with higher shock intensity. It is because the increasing number of shocks makes the system more vulnerable to hard failures. For example, if the braking system is used very frequently, a large among of heat is generated by the friction between the brake pads and brake disc, then the braking system is more likely to fail because thermal shocks can cause a sudden reduction of the friction coefficient of brake pads.

5. Conclusion

In previous degradation-based work on system reliability, the components in one system are typically assumed to suffer from similar CFPs, neglecting their failure mechanisms. In this work, a new generalized reliability model is proposed for systems subject to multiple degradation and shock processes, including pure DPs without CFPs, independent and dependent CFPs. The new model is presented in terms of the generalized TED process, which allows the model to be applied to systems whose components degrade with different stochastic processes, such as the Wiener, Gamma, inverse Gaussian, and other processes. Furthermore, compared with the existing degradation-based reliability methods, which mainly focus on series and parallel systems, the combination of the survival signature enables the application of the proposed method to systems with general structures, including series, parallel, bridge, and network structural systems. An automotive braking system with four types of components, which are exposed to multiple degradation and shock processes, is applied as a numerical example to illustrate the application of the proposed method, and the reliability of this braking system is analyzed by the proposed method in different scenarios, including single and multiple degradation and shock processes. The differences among the reliability results show the necessity of considering the variety of the DPs and CFPs of multi-component systems.

In this work, a generalized reliability model is proposed, which is applicable to various structural systems with multiple DPs and CFPs, but the systems are assumed to be unrepairable. Therefore, it would be an interesting problem to extend the generalized reliability model to systems with repairable components. One kind of dependence between the degradation and shock processes is considered in this work. It is worthwhile to investigate more generalized reliability methods for systems subject to other dependent CFPs. In addition, random shocks are modelled by a homogeneous process, the presented work can be extended by considering a shock process with changing intensity, such as being a function of time or related to the degradation levels of the components.

Acknowledgements

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Appendix A

If the degradation value of the component of type k is less than L_k, then the probability that the component survives in the first case is:

$$\begin{aligned} P\left(A_k B_{1k}\right) &= \sum_{n_k=0}^{\infty} P\left(Xs_k\left(t\right) < L_k, N_k\left(t\right) = n_k, \bigcap\nolimits_{j_k=1}^{n_k}\left(W_{j_k} < D_{1k}\right)\right) \\ &= \sum_{n_k=0}^{\infty} P\left(X_k\left(t\right) + S_k\left(t\right) < L_k \middle| N_k\left(t\right) = n_k\right) P\left(N_k\left(t\right) = n_k\right) \prod_{j_k=1}^{n_k} P\left(W_{j_k} < D_{1k}\right) \end{aligned} \tag{A.1}$$

If the degradation value of the component of type k is no less than L_k, then the probability that the component survives in the second case is:

$$\begin{aligned} P\left(A_k B_{2k}\right) = & \sum_{n_k=0}^{\infty} \sum_{n_{1k}=0}^{n_k} P\Big(L_k \le Xs_k\left(t\right) < H_k, N_k\left(t_{*k}\right) = n_{1k}, N_k\left(t - t_{*k}\right) = n_k - n_{1k}, \\ & \bigcap\nolimits_{j_k=1}^{n_{1k}}\left(W_{j_k} < D_{1k}\right), \bigcap\nolimits_{j_k=n_{1k}+1}^{n_k}\left(W_{j_k} < D_{2k}\right)\Big) \\ & = P\left(L_k \le X_k\left(t\right) < H_k \middle| N_k\left(t=0\right)\right) P\left(N_k\left(t\right) = 0\right) \\ & + \sum_{n_k=1}^{\infty} \sum_{n_{1k}=0}^{n_k} \int_0^t P\left(L_k \le X_k\left(t\right) + S_k\left(t\right) < H_k \middle| N_k\left(t\right) = n_k\right) P\left(N_k\left(t_{*k}\right) = n_{1k}\right) \\ & \cdot P\left(N_k\left(t - t_{*k}\right) = n_k - n_{1k}\right) \prod_{j_k=1}^{n_{1k}} P\left(W_{j_k} < D_{1k}\right) \prod_{j_k=n_{1k}+1}^{n_k} P\left(W_{j_k} < D_{2k}\right) \\ & \cdot f\left(t_{*k} \middle| N_k\left(t_{*k}\right) = n_{1k}\right) dt_{*k} \end{aligned} \tag{A.2}$$

where t_{*k} is the time when the degradation value of the component reaches L_k, $f(t_{*k}|N_k(t_{*k}) = n_{1k})$ is the conditional PDF of t_{*k} when the number of shocks on the component is n_{1k}, expressed as:

$$f\left(t_{*k}\left|N\left(t_{*k}\right)=n_{1k}\right.\right)=\frac{\partial F\left(t_{*k}\left|N\left(t_{*k}\right)=n_{1k}\right.\right)}{\partial t_{*k}} \tag{A.3}$$

where $F(t_{*k}|N_k(t_{*k}) = n_{1k})$ is the conditional CDF of t_{*k}, expressed as:

$$\begin{aligned}F\left(t_{*k}\left|N_k\left(t_{*k}\right)=n_{1k}\right.\right)&=P\left(T\le t_{*k}\left|N_k\left(t_{*k}\right)=n_{1k}\right.\right)\\&=P\left(Xs_k\left(t_{*k}\right)\ge L_k\left|N_k\left(t_{*}\right)=n_{1k}\right.\right)\\&=1-P\left(Xs_k\left(t_{*k}\right)\ge L_k\left|N_k\left(t_{*}\right)=n_{1k}\right.\right)\end{aligned} \tag{A.4}$$

where

$$\begin{aligned}P\left(Xs_k\left(t_{*k}\right)\ge L_k\left|N_k\left(t_{*k}\right)=n_{1k}\right.\right)&=\begin{cases}P\left(X_k\left(t_{*k}\right)+S_k\left(t_{*k}\right)\le L_k\left|N_k\left(t_{*k}\right)=n_{1k},n_{1k}\ne 0\right.\right)\\P\left(X_k\left(t_{*}\right)\le L_k\left|N_k\left(t_{*k}\right)=n_{1k},n_{1k}=0\right.\right)\end{cases}\\&=\begin{cases}\left[\int_0^{L_k}\int_0^{L_k-x}g_{S_k(t)}\left(u\left|N_k\left(t_{*k}\right)=n_{1k},n_{1k}\ne 0\right.\right)\right.\\\left.\cdot f_{X_k(t)}\left(x_k\left|\mu_k,\lambda_k,p_k,t_{*k}\right.\right)dudx_k\right]\\\int_0^{L_k}f_{X_k(t)}\left(x_k\left|\mu_k,\lambda_k,p_k,t_{*k}\right.\right)dx_k\end{cases}\end{aligned}$$

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