Nonparametric predictive inference for competing risks

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Abstract

In reliability, failure data often correspond to competing risks, where several failure modes can cause a unit to fail. This paper presents nonparametric predictive inference (NPI) for competing risks data, assuming that the different failure modes are independent. NPI is a statistical approach based on few assumptions, with inferences strongly based on data and with uncertainty quantified via lower and upper probabilities. The focus is on the lower and upper probabilities for the event that a future unit will fail due to a specific failure mode. The paper illustrates the effect of grouping different failure modes together, and some special cases and features are discussed. It is also shown that NPI can easily deal with competing risks data resulting from experiments with progressive censoring. Furthermore, new formulae are presented for the NPI lower and upper survival functions.

Key words: competing risks; imprecise reliability; lower and upper probabilities; lower and upper survival functions; nonparametric predictive inference; progressive censoring; right-censored data.

1 Introduction

In reliability, failure data often correspond to competing risks [1, 2], where several failure modes can cause a unit to fail, and where failure occurs due to the first failure event caused by one of the failure modes. Throughout this paper, it is assumed that each unit cannot fail more than once and it is not used any further once it has failed, and that a failure is caused by a single failure mode which, upon observing a failure, is known with certainty.

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Tsiatis [3] showed that failure data resulting from such competing risks cannot be used to identify dependence between the failure modes. Effectively, this means that such data can only be used to learn about the marginal distributions, which are the distributions of failure times restricted to single failure modes, for which all failures caused by other failure modes lead to right-censored observations. Throughout this paper it is assumed that the failure modes are independent, inclusion of assumed dependence would be an interesting topic for future research, but cannot be learned about from the data as considered here as shown by Tsiatis, and nonparametric predictive inference (NPI) has not yet been developed to take such dependence into account.

NPI [4, 5] is a statistical approach based on few assumptions, with inferences strongly based on data and uncertainty quantified via lower and upper probabilities. NPI is based on Hill's assumption $A_{(n)}$ [6, 7, 8], which gives a direct probability for a future observable random quantity, conditional on observed values of related random quantities. In this paper, NPI lower and upper probabilities are presented for the event that a future unit, say unit $n+1$, will fail due to a specific failure mode, based on data consisting of times of failures resulting from competing risks for n units. This approach uses NPI for right-censored data as presented by Coolen and Yan [9]. The use of lower and upper probabilities to quantify uncertainty has gained increasing attention during the last decade, short and detailed overviews of theories and applications in reliability, together called 'imprecise reliability', are presented by Coolen and Utkin [10, 11].

Section 2 of this paper presents a brief overview of NPI, including explanation of how NPI deals with right-censored data. New expressions for NPI lower and upper survival functions are also presented. Section 3 presents NPI for the competing risks problem, which is illustrated by some examples in Section 4. NPI can also be applied for different censoring mechanisms, which is illustrated in Section 5 for competing risks inferences under progressive censoring. Some concluding remarks are given in Section 6. The paper finishes with appendices including a list of notation used and proofs of main results in the paper.

2 Nonparametric Predictive Inference

Nonparametric predictive inference (NPI) is a statistical method based on Hill's assumption $A_{(n)}$ [6, 7, 8], which gives a direct conditional probability for a future observable random quantity, conditional on observed values of related random quantities [4, 5]. Let $Y_1, \ldots, Y_n, Y_{n+1}$ be positive, continuous and exchangeable random quantities representing event times. Suppose that the values of $Y_1, \ldots, Y_n$ are observed and the corresponding ordered observed values are denoted by $0 < y_1 < \ldots < y_n < \infty$, for ease of notation let $y_0 = 0$. For ease of presentation, it is assumed that no ties occur among the observed values. However, it is quite straightforward to deal with tied observations in this setting, by assuming that tied observations differ by small amounts which tend to zero. For the random quantity

Y_{n+1} representing a future observation, based on n observations, the assumption $A_{(n)}$ [6] is

$$P(Y_{n+1} \in (y_{i-1}, y_i)) = \frac{1}{n+1},\ i = 1, \ldots, n,\ \text{and}\ P(Y_{n+1} \in (y_n, \infty)) = \frac{1}{n+1}$$

$A_{(n)}$ does not assume anything else, and can be interpreted as a post-data assumption related to exchangeability [12], a detailed discussion of $A_{(n)}$ is provided by Hill [7]. Inferences based on $A_{(n)}$ are predictive and nonparametric, and can be considered suitable if there is hardly any knowledge about the random quantity of interest, other than the n observations, or if one does not want to use such information, e.g. to study effects of additional assumptions underlying other statistical methods. $A_{(n)}$ is not sufficient to derive precise probabilities for many events of interest, but it provides bounds for probabilities via the 'fundamental theorem of probability' [12], which are lower and upper probabilities in interval probability theory [4, 13, 14, 15].

In reliability and survival analysis, data on event times are often affected by right-censoring, where for a specific unit or individual it is only known that the event has not yet taken place at a specific time. Coolen and Yan [9] presented a generalization of $A_{(n)}$, called 'right-censoring $A_{(n)}$' or rc-$A_{(n)}$, which is suitable for right-censored data. In comparison to $A_{(n)}$, rc-$A_{(n)}$ uses the additional assumption that, at the moment of censoring, the residual lifetime of a right-censored unit is exchangeable with the residual lifetimes of all other units that have not yet failed or been censored, see [9, 16] for further details of rc-$A_{(n)}$. Coolen, *et al* [17] introduced NPI to some reliability applications, including upper and lower survival functions for the next future observation, illustrated with an application with competing risks data. They illustrated the upper and lower marginal survival functions, so each restricted to a single failure mode. In this paper, the main question considered is which failure mode will cause the next unit to fail, or for example in survival analysis terminology, which disease causes the next individual considered to die. From now on, terminology from reliability will be used, so events considered are failures of units, but the methods proposed are of course more generally applicable.

To formulate the required form of rc-$A_{(n)}$ for NPI with competing risks, notation is required for probability mass assigned to intervals without further restrictions on the spread within the intervals. Such a partial specification of a probability distribution is called an M-function [9], which is mathematically equivalent to Shafer's 'basic probability assignments' [18], and is given by the following definition.

Definition: M-Function

A partial specification of a probability distribution for a real-valued random quantity Y can be provided via probability masses assigned to intervals, without any further restriction on the spread of the probability mass within each interval. A probability mass assigned, in such a way, to an interval (a, b) is denoted by $M_Y(a, b)$ and referred to as M-function value for Y on (a, b). Clearly, all M-function values for Y on all intervals should sum up to one, each

M-function value should be in [0,1] and $A_{(n)}$ can be expressed as $M_{Y_{n+1}}(y_i, y_{i+1}) = 1/(n+1)$, for $i = 0, \ldots, n-1$, and $M_{Y_{n+1}}(y_n, \infty) = 1/(n+1)$.

Suppose that there are n observations consisting of u failure times, $x_1 < x_2 < \ldots < x_u$, and $\upsilon(= n-u)$ right-censored observations, $c_1 < c_2 < \ldots < c_\upsilon$. Let $x_0 = 0$ and $x_{u+1} = \infty$. Suppose further that there are s_i right-censored observations in the interval (x_i, x_{i+1}), denoted by $c_1^i < c_2^i < \ldots < c_{s_i}^i$, so $\sum_{i=0}^{u} s_i = \upsilon$. The assumption rc-$A_{(n)}$ partially specifies the NPI-based probability distribution for X_{n+1} by the following M-function values, where the random quantity X_{n+1} represents the failure time of one future unit [9],

$$M_{X_{n+1}}(t_{i^*}^i, x_{i+1}) = \frac{1}{(n+1)}(\tilde{n}_{t_{i^*}^i})^{\delta_{i^*}^i - 1} \prod_{\{r: c_r < t_{i^*}^i\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} \quad ; \quad i = 0, 1, \ldots, u \text{ and } i^* = 0, 1, \ldots, s_i \tag{1}$$

where

$$\delta_{i^*}^i = \begin{cases} 1 & \text{if } i^* = 0 \quad \text{i.e. } t_0^i = x_i \quad \text{(failure time or time 0)} \\ 0 & \text{if } i^* = 1, \ldots, s_i \quad \text{i.e. } t_{i^*}^i = c_{i^*}^i \quad \text{(censoring time)} \end{cases}$$

and $\tilde{n}_{c_r}$ and $\tilde{n}_{t_{i^*}^i}$ are the number of units in the risk set just prior to time c_r and $t_{i^*}^i$, respectively. For consistency of notation, the further definition $\tilde{n}_0 = n+1$ is used throughout this paper. Only intervals of this form have positive M-function values, and these sum up to one over all these intervals. Summing up all M-function values assigned to such intervals with the same x_{i+1} as right endpoint gives the probability

$$P(X_{n+1} \in (x_i, x_{i+1})) = \frac{1}{n+1} \prod_{\{r: c_r < x_{i+1}\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} \tag{2}$$

where x_i and x_{i+1} are two sequential failure times (and $x_0 = 0$, $x_{u+1} = \infty$). It should be noted that, throughout this paper, the product taken over an empty set is equal to one.

Before considering NPI for competing risks, it is worth to present new formulae for the NPI lower and upper survival functions, $\underline{S}_{X_{n+1}}(t)$ and $\overline{S}_{X_{n+1}}(t)$, respectively, as first introduced by Coolen, *et al* [17], which are used in Appendix C. These formulae are the simplest closed-form expressions for these lower and upper survival functions presented in the literature thus far, and as such are likely to be useful in many more applications of NPI in reliability and survival analysis. In addition to notation introduced above, let $t_{s_i+1}^i = t_0^{i+1} = x_{i+1}$ for $i = 0, 1, \ldots, u-1$. The NPI lower survival function [17] can be expressed as follows, for $t \in [t_a^i, t_{a+1}^i)$ with $i = 0, 1, \ldots, u$ and $a = 0, 1, \ldots, s_i$,

$$\underline{S}_{X_{n+1}}(t) = \frac{1}{n+1}\, \tilde{n}_{t_a^i} \prod_{\{r: c_r < t_a^i\}} \frac{\tilde{n}_{c_r}+1}{\tilde{n}_{c_r}} \tag{3}$$

and the corresponding NPI upper survival function [17] can be written as follows, for $t \in$

$[x_i, x_{i+1})$ with $i = 0, 1, \ldots, u$,

$$\overline{S}_{X_{n+1}}(t) = \frac{1}{n+1}\, \tilde{n}_{x_i} \prod_{\{r: c_r < x_i\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} \tag{4}$$

The proof of these new formulae are given in Appendix A.

Coolen and Yan [19] presented NPI for comparison of two groups of lifetime data including right-censored observations. By applying rc-$A_{(n)}$ for each group, the NPI method for comparison of groups is based on the direct comparison of the next future observation from each group. Maturi, *et al* [20] extend this for comparing more than two groups in order to select the best group, in terms of largest lifetime. Also Maturi, *et al* [21] consider selection of subsets of the groups according to several criteria. They allow early termination of the experiment in order to save time and cost, which effectively means that all units in all groups that have not yet failed are right-censored at the time the experiment is ended.

3 NPI for Competing Risks

This paper considers competing risks, with k distinct failure modes that can cause a unit to fail. It is assumed that the unit fails due to the first occurrence of a failure mode, and that the unit is withdrawn from further use and observation at that moment. It is further assumed that such failure observations are obtained for n units, and that the failure mode causing a failure is known with certainty. As is common in study of failure data under competing risks, for each unit k random quantities are considered, say T_j for $j = 1, \ldots, k$, where T_j represents the unit's time to failure under the condition that failure occurs due to failure mode j. These T_j are assumed to be independent continuous random quantities, which implies the assumption that the failure modes occur independently, and the failure time of the unit is $T = \min(T_1, \ldots, T_k)$. Therefore, each unit considered can have one failure time and it will be known with certainty which failure mode caused a failure. Hence, for the T_j corresponding to the other failure modes, which did not cause the failure of the unit, the unit's observed failure time is a right-censoring time.

For the NPI approach, let the failure time of a future unit be denoted by X_{n+1}, and let the corresponding notation for the failure time including indication of the actual failure mode, say failure mode j, be $X_{j,n+1}$ (so X_{n+1} corresponds to an observation T for unit $n+1$, and $X_{j,n+1}$ to T_j, according to the notation in the previous paragraph). As the different failure modes are assumed to occur independently, the competing risk data per failure mode consist of a number of observed failure times for failures caused by the specific failure mode considered, and right-censoring times for failures caused by other failure modes. Hence rc-$A_{(n)}$ can be applied per failure mode j, for inference on $X_{j,n+1}$. Let the number of failures caused by failure mode j be u_j, $x_{j,1} < x_{j,2} < \ldots, < x_{j,u_j}$, and let $v_j (= n - u_j)$ be the number of the right-censored observations, $c_{j,1} < c_{j,2} < \ldots < c_{j,v_j}$, corresponding to failure mode j. For notational convenience, let $x_{j,0} = 0$ and $x_{j,u_j+1} = \infty$. Suppose further that there are s_{j,i_j} right-censored observations in the interval (x_{j,i_j}, x_{j,i_j+1}), denoted by $c^{i_j}_{j,1} < c^{i_j}_{j,2} < \ldots < c^{i_j}_{j,s_{j,i_j}}$, so $\sum_{i_j=0}^{u_j} s_{j,i_j} = \upsilon_j$. It should be emphasized that it is not assumed that each unit considered must actually fail, if a unit does not fail then there will be a right-censored observation recorded for this unit for each failure mode, as it is assumed that the unit will then be withdrawn from the study, or the study ends, at some point. The random quantity representing the failure time of the next unit, with all k failure modes considered, is $X_{n+1} = \min\limits_{1\le j\le k} X_{j,n+1}$.

The NPI M-functions for $X_{j,n+1}$ ($j = 1, \ldots, k$), similar to (1), are

$$M^j(t^{i_j}_{j,i^*_j}, x_{j,i_j+1}) = M_{X_{j,n_j+1}}(t^{i_j}_{j,i^*_j}, x_{j,i_j+1}) = \frac{1}{(n+1)}(\tilde{n}_{t^{i_j}_{j,i^*_j}})^{\delta^{i_j}_{i^*_j}-1} \prod_{\{r:c_{j,r}<t^{i_j}_{j,i^*_j}\}} \frac{\tilde{n}_{c_{j,r}}+1}{\tilde{n}_{c_{j,r}}} \qquad (5)$$

where $i_j = 0, 1, \ldots, u_j$, $i^*_j = 0, 1, \ldots, s_{j,i_j}$ and

$$\delta^{i_j}_{i^*_j} = \begin{cases} 1 & \text{if } i^*_j = 0 \quad \text{i.e. } t^{i_j}_{j,0} = x_{j,i_j} \quad \text{(failure time or time 0)} \\ 0 & \text{if } i^*_j = 1, \ldots, s_{j,i_j} \quad \text{i.e. } t^{i_j}_{j,i^*_j} = c^{i_j}_{j,i^*_j} \quad \text{(censoring time)} \end{cases}$$

Again $\tilde{n}_{c_r}$ and $\tilde{n}_{t^{i_j}_{j,i^*_j}}$ are the numbers of units in the risk set just prior to times c_r and $t^{i_j}_{j,i^*_j}$, respectively. The corresponding NPI probabilities, similar to (2), are

$$P^j(x_{j,i_j}, x_{j,i_j+1}) = P(X_{j,n_j+1} \in (x_{j,i_j}, x_{j,i_j+1})) = \frac{1}{n+1} \prod_{\{r:c_{j,r}<x_{j,i_j+1}\}} \frac{\tilde{n}_{c_{j,r}}+1}{\tilde{n}_{c_{j,r}}} \qquad (6)$$

where x_{j,i_j} and x_{j,i_j+1} are two consecutive observed failure times caused by failure mode j (and $x_{j,0} = 0$, $x_{j,u_j+1} = \infty$).

In this paper, the main event of interest is that a single future unit, called the 'next unit', undergoing the same test or process as the n units for which failure data are available, fails due to a specific failure mode, say mode l. NPI lower and upper probabilities for this event are derived, for each $l = 1, \ldots, k$. The following notation is used for these NPI lower and upper probabilities, respectively, for the event of interest

$$\underline{P}^{(l)} = \underline{P}\left(X_{l,n+1} = \min_{1\le j\le k} X_{j,n+1}\right) = \underline{P}\left(X_{l,n+1} < \min_{\substack{1\le j\le k\\ j\ne l}} X_{j,n+1}\right)$$

$$\overline{P}^{(l)} = \overline{P}\left(X_{l,n+1} = \min_{1\le j\le k} X_{j,n+1}\right) = \overline{P}\left(X_{l,n+1} < \min_{\substack{1\le j\le k\\ j\ne l}} X_{j,n+1}\right)$$

These NPI lower and upper probabilities for the event that the next unit will fail due to failure mode l are

$$\underline{P}^{(l)} = \sum_{C(j,\, i_j,\, i_j^*)} \left[\sum_{i_l=0}^{u_l} 1(x_{l,i_l+1} < \min_{\substack{1\le j\le k\\ j\neq l}} \{t_{j,i_j^*}^{i_j}\}) P^l(x_{l,i_l}, x_{l,i_l+1}) \right] \prod_{\substack{j=1\\ j\neq l}}^{k} M^j(t_{j,i_j^*}^{i_j}, x_{j,i_j+1}) \tag{7}$$

$$\overline{P}^{(l)} = \sum_{C(j,\, i_j)} \left[\sum_{i_l=0}^{u_l} \sum_{i_l^*=0}^{s_{l,i_l}} 1(t_{l,i_l^*}^{i_l} < \min_{\substack{1\le j\le k\\ j\neq l}} \{x_{j,i_j+1}\}) M^l(t_{l,i_l^*}^{i_l}, x_{l,i_l+1}) \right] \prod_{\substack{j=1\\ j\neq l}}^{k} P^j(x_{j,i_j}, x_{j,i_j+1}) \tag{8}$$

where $\sum_{C(j,\, i_j,\, i_j^*)}$ denotes the sums over all i_j^* from 0 to s_{j,i_j} and over all i_j from 0 to u_j for $j = 1, \ldots, k$ but not including $j = l$. Similarly, $\sum_{C(j,\, i_j)}$ denotes the sums over all i_j from 0 to u_j for $j = 1, \ldots, k$ but not including $j = l$. The derivation of these NPI lower and upper probabilities is given in Appendix B.

Before illustrating and discussing this method in examples in Section 4, it is of interest to consider the special case of the competing risks problem in which there are only two failure modes (so $k = 2$), say modes l and j, and in which each of the n units considered actually fails due to one of these two failure modes. Therefore, any unit which fails due to failure mode l leads to a right-censored observation for failure mode j, and vice versa. In this case, the number of failures due to failure mode l (j) is equal to the number of right-censored observations for failure mode j (l), so $v_l = u_j$ and $v_j = u_l$. Let R_l (R_j) be the set of ranks of all ordered failure times due to failure mode l (j), so $R_l \subset \{1, 2, \ldots, n\}$ and $R_j = \{1, 2, \ldots, n\} \setminus R_l$. The NPI lower and upper probabilities (7) and (8) for the event that the next unit will fail due to failure mode l are then

$$\underline{P}^{(l)} = \frac{1}{n+1} \sum_{r_l \in R_l} \frac{\tilde{n}_{x_{l,(r_l)}}}{\tilde{n}_{x_{l,(r_l)}} + 1} = \frac{1}{n+1} \sum_{r_l \in R_l} \frac{n+1-r_l}{n+2-r_l} \tag{9}$$

$$\overline{P}^{(l)} = 1 - \frac{1}{n+1} \sum_{r_j \in R_j} \frac{\tilde{n}_{x_{j,(r_j)}}}{\tilde{n}_{x_{j,(r_j)}} + 1} = 1 - \frac{1}{n+1} \sum_{r_j \in R_j} \frac{n+1-r_j}{n+2-r_j} \tag{10}$$

The derivation of these NPI lower and upper probabilities is given in Appendix C (which uses a result derived in Appendix A).

The formulae (9) and (10) enable the derivation of some interesting results of the NPI approach in this specific setting, with only two failure modes and all n units actually having failed. Consider the following two specific scenarios in detail:

(A) all failures due to failure mode j come first, followed by all failures from failure mode l, meaning that the u_j failure times of failures due to mode j are all smaller than the u_l failure times of failures due to mode l. In this case, the NPI lower and upper probabilities for the event that the next unit will fail due to failure mode l are

$$\underline{P}^{(l),\,\mathrm{A}} = \frac{1}{n+1}\sum_{i=1}^{u_l}\frac{i}{i+1}$$

$$\overline{P}^{(l),\,\mathrm{A}} = 1 - \frac{1}{n+1}\sum_{i=u_l+1}^{n}\frac{i}{i+1}$$

(B) all failures due to failure mode l come first, followed by all failures from failure mode j, in which case the NPI lower and upper probabilities for the event of interest are

$$\underline{P}^{(l),\,\mathrm{B}} = \frac{1}{n+1}\sum_{i=u_j+1}^{n}\frac{i}{i+1}$$

$$\overline{P}^{(l),\,\mathrm{B}} = 1 - \frac{1}{n+1}\sum_{i=1}^{u_j}\frac{i}{i+1}$$

These NPI lower and upper probabilities follow straightforwardly from the general expressions (9) and (10) for these two special cases. Because $\frac{i}{i+1}$ is increasing in i, these results imply that case (A) leads to the minimal NPI lower and upper probabilities when all possible orderings of u_j failures due to mode j and u_l failures due to mode l are considered, while case (B) leads to the maximal NPI lower and upper probabilities in this setting. More generally, these results imply a nice monotonicity result, namely that the NPI lower and upper probabilities (9) and (10) increase whenever any failure caused by failure mode l would move to an earlier place in the ordering. This is illustrated in Example 1 in Section 4.

The difference between the corresponding NPI upper and lower probabilities for an event of interest is called the 'imprecision', it has been suggested that it relates quite logically to the amount of information available [13]. The imprecision for the event considered here, in this special case of competing risks with only two failure modes and all n units actually having failed, does not depend on the number of failures caused by each failure mode nor on their ordering, and it is equal to

$$\text{Imprecision} = \overline{P}^{(l)} - \underline{P}^{(l)} = \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{i}$$

The proof of this nice feature is given in Appendix C. It should be emphasized that these attractive properties of the NPI results in the case of two competing risks do not generalize to more than two competing risks, due to the fact that the product terms in the NPI lower and upper probabilities (7) and (8) only disappear for this case with two failure modes and all n units actually having failed.

4 Examples

In this section three examples of NPI for competing risks are presented to illustrate the method and to discuss some of its properties. Example 1 is a small example which serves to illustrate the results in the case with only two failure modes as discussed at the end of Section 3. Examples 2 and 3 involve a substantial competing risks data set from the literature, with different groupings of failure modes and also including some units which did not fail at all during the study, hence leading to right-censored observations for each of the failure modes considered. This will illustrate a further important aspect of NPI in this setting, and will also lead to a conjecture.

Example 1

Consider an experiment in which five units are subjected to two failure modes, FM1 and FM2, which are competing risks in the manner discussed in this paper. Suppose that all five units are observed to fail, and that three units fail due to FM1 and two units due to FM2. So the failure times of the three units failing due to FM1 are right-censored observations for FM2, and the failure times of the two units which fail due to FM2 are right-censored observations for FM1. As all five units actually fail during the experiment, no further right-censored observations occur in this example. Suppose that there had actually been a sixth unit in the experiment, and this was randomly selected before the start of the experiment as the unit for which the failure information would not be revealed to us. The method presented in this paper provides inferences for the event that this sixth unit fails due to FM1 or due to FM2 (instead 'will fail' could be used, if the inferences are interpreted as involving a future unit undergoing the same experiment, both are convenient ways to think about the setting and inferences).

In this NPI approach, the actual failure times of the five units are not important, only their ordering with regard to failure modes is important. Of course, NPI can also be used for inference on the actual failure time of the sixth unit, for example by considering the event that this unit will not fail before a specified time, in which case the failure times of the five units are explicitly used, not only their ordering with regard to the failure modes, this is briefly illustrated for Examples 2 and 3 at the end of this section. There are 10 possible orderings for the failure modes FM1 and FM2, with three units failing due to FM1 and two due to FM2. The lower and upper probabilities that the sixth unit fails due to FM1, for the ten possible orderings of the two failure modes, are given in Table 1.

Consider the ordering O_1, in which the three failures due to FM1 happen before the two failures caused by FM2, and which corresponds to case (B) discussed in Section 3. The NPI lower and upper probabilities for the event that the sixth unit fails due to FM1 are, for this ordering O_1, greater than the corresponding lower and upper probabilities for all other orderings of the failure modes. On the other hand, ordering O_{10}, in which the two failures due to FM2 happen before the three failures caused by FM1, and which corresponds to case (A) in Section 3, leads to the minimum lower and upper probabilities, over all orderings, for the event that the sixth unit will fail due to FM1. Table 1 also illustrates the monotonicity result mentioned in Section 3, namely that the NPI lower and upper probabilities for the

	FM Orderings					$\underline{P}(X_6^{FM1} < X_6^{FM2})$	$\overline{P}(X_6^{FM1} < X_6^{FM2})$
O_1	1	1	1	2	2	0.3972	0.8056
O_2	1	1	2	1	2	0.3833	0.7917
O_3	1	1	2	2	1	0.3556	0.7639
O_4	1	2	1	1	2	0.3750	0.7833
O_5	1	2	1	2	1	0.3472	0.7556
O_6	1	2	2	1	1	0.3333	0.7417
O_7	2	1	1	1	2	0.3694	0.7778
O_8	2	1	1	2	1	0.3417	0.7500
O_9	2	1	2	1	1	0.3278	0.7361
O_{10}	2	2	1	1	1	0.3194	0.7278

Table 1: Lower and upper probabilities for the sixth unit to fail due to FM1

next unit to fail due to FM1 increase if any failure caused by FM1 moves to an earlier place in the ordering.

The NPI lower and upper probabilities for the event that the sixth unit fails due to FM2, for the different orderings of the failure modes for the data, follow from those for FM1 reported in Table 1 by the conjugacy property [4, 13],

$$\underline{P}(X_6^{FM2} < X_6^{FM1}) = 1 - \overline{P}(X_6^{FM1} < X_6^{FM2})$$

$$\overline{P}(X_6^{FM2} < X_6^{FM1}) = 1 - \underline{P}(X_6^{FM1} < X_6^{FM2})$$

Consider, for example, the ordering O_5, for which the corresponding NPI lower and upper probabilities that the sixth unit fails due to FM1 are 0.3472 and 0.7556, while for this unit to fail due to FM2 they are 0.2444 and 0.6528. On the basis of these NPI lower and upper probabilities alone, one could conclude that there is a weak indication that failure due to FM1 is more likely than due to FM2, as

$$\underline{P}(X_6^{FM1} < X_6^{FM2}) = 0.3472 > 0.2444 = \underline{P}(X_6^{FM2} < X_6^{FM1})$$

and

$$\overline{P}(X_6^{FM1} < X_6^{FM2}) = 0.7556 > 0.6528 = \overline{P}(X_6^{FM2} < X_6^{FM1})$$

One could speak about a strong indication for the event that failure of the sixth unit will be caused by FM1 if $\underline{P}(X_6^{FM1} < X_6^{FM2}) > \overline{P}(X_6^{FM2} < X_6^{FM1})$, which does not occur for any of the orderings in this example. Finally, the imprecision in this example, for all orderings of the two failure modes, is equal to 0.4084, illustrating the final property presented in Section 3.

Example 2

In this example and in Example 3, a well-known data set from the literature [22] is used to illustrate some aspects of the NPI method for dealing with competing risks. The data contain information about 36 units of a new model of a small electrical appliance which were tested, and where the lifetime observation per unit consists of the number of completed cycles of use until the unit failed. These data are presented in Table 2, which also includes the specific failure mode (FM) that caused the unit to fail. In the study, there were 18 different ways in which an appliance could fail, so 18 failure modes, but to illustrate the NPI method this number is reduce to two (groups of) failure modes in the current example, while grouping into three failure modes is considered in Example 3, after which the differences between these examples are discussed. Three units in the test did not fail before the end of the experiment, so for these units right-censored observations (2565, 6367 and 13403) are recorded for all failure modes considered, indicated by '-' for the failure mode in Table 2.

# cycles	FM	# cycles	FM	# cycles	FM
11	1	1990	9	3034	9
35	15	2223	9	3034	9
49	15	2327	6	3059	6
170	6	2400	9	3112	9
329	6	2451	5	3214	9
381	6	2471	9	3478	9
708	6	2551	9	3504	9
958	10	2565	-	4329	9
1062	5	2568	9	6367	-
1167	9	2702	10	6976	9
1594	2	2761	6	7846	9
1925	9	2831	2	13403	-

Table 2: Failure data for electrical appliance test

The two most frequently occurring failure modes in these data are FM9, which caused 17 units to fail, and FM6 which caused 7 failures. It is considered how likely it is that the next unit, say unit 37, would fail due to FM9, assuming it would undergo the same test and its number of completed cycles would be exchangeable with these numbers for the 36 units reported. In this example, all failure modes other than FM9 are grouped together, and these are jointly considered as a single failure mode, which enables illustration of the NPI approach with 2 failure modes, FM9 and, say, 'other failure mode' (OFM). There are still three units that do not fail, and hence for which there are only right-censored observations (RC). For clarity, the data corresponding to this definition of failure modes are presented in Table 3.

When the theory for NPI for competing risks data was presented in Section 3, it was assumed that there were no ties to avoid notational difficulties. In this example, however, there are tied observations, as two units have failed after 3034 completed cycles, both failed due to FM9. To deal with this, it is assumed that there is a small difference between these

FM9	1167	1925	1990	2223	2400	2471	2551	2568	3034	3034
	3112	3214	3478	3504	4329	6976	7846			
OFM	11	35	49	170	329	381	708	958	1062	1594
	2327	2451	2702	2761	2831	3059				
RC	2565	6367	13403							

Table 3: Failure data for electrical appliance test: FM9, OFM and RC

values, such that their ordering does not change with regard to observations of units in other groups. It is actually assumed that one of these two units failed after 3035 completed cycles. If such a tie would occur among different groups, then one can break it similarly in two ways, different for upper and lower probabilities in such a way that these are maximal and minimal, respectively, over the possible ways of breaking such ties, without changing the order of these observations with respect to all other observations. Implicit in the NPI method for competing risks data is that a failure time observation caused by one failure mode is simultaneously a right-censored observation for all other failure modes. This situation is dealt with in the NPI approach, as is common in many statistical approaches, by assuming that the right-censoring time is just beyond the failure time. The three right-censored observations, for units that were not observed to fail during the experiment, also lead to tied observations for the two failure modes (FM9 and OFM) considered, as for both the right-censoring times coincide. This is also dealt with by assuming that for one of the failure modes this event occurred fractionally later than for the other failure mode, and then the lower and upper probabilities for the event of interest are again calculated by considering the maximum and minimum of the upper and lower probabilities, respectively, corresponding to the different possible orderings of these 'un-tied' right-censoring times.

The NPI lower and upper probabilities for the event that unit 37 will fail due to FM9 are

$$\underline{P}(X_{37}^{FM9} < X_{37}^{OFM}) = 0.4358, \quad \overline{P}(X_{37}^{FM9} < X_{37}^{OFM}) = 0.5804$$

while the corresponding NPI lower and upper probabilities for unit 37 to fail due to OFM are

$$\underline{P}(X_{37}^{OFM} < X_{37}^{FM9}) = 0.4196, \quad \overline{P}(X_{37}^{OFM} < X_{37}^{FM9}) = 0.5642$$

These lower and upper probabilities satisfy the conjugacy property [4, 13], which is due to the fact that, implicit in our method, it is assumed that the experiment on unit 37 would actually continue until it fails, and this is assumed to happen with certainty. On the basis of these NPI lower and upper probabilities, the data could be considered to contain a weak indication that the event that unit 37 will fail due to FM9 is a bit more likely than for it to fail due to another failure mode, with all the other failure modes grouped together as done in this example.

Example 3

This example uses the same data as Example 2, but the failure modes are grouped differently. Both FM9 and FM6 are considered separately, with 17 and 7 units that failed due to them, respectively, and all other failure modes are grouped into one 'other failure mode' (OFM). For clarity, the data used here are given in Table 4.

FM9	1167	1925	1990	2223	2400	2471	2551	2568	3034	3034
	3112	3214	3478	3504	4329	6976	7846			
FM6	170	329	381	708	2327	2761	3059			
OFM	11	35	49	958	1062	1594	2451	2702	2831	
RC	2565	6367	13403							

Table 4: Failure data for electrical appliance test: FM9, FM6, OFM and RC

The NPI lower and upper probabilities for the event that unit 37 will fail due to FM9, due to FM6 or due to OFM, are as follows

$$\underline{P}\Big(X_{37}^{FM9}<\min\big\{X_{37}^{FM6},X_{37}^{OFM}\big\}\Big)=0.3915\ ,\ \overline{P}\Big(X_{37}^{FM9}<\min\big\{X_{37}^{FM6},X_{37}^{OFM}\big\}\Big)=0.5804$$

$$\underline{P}\Big(X_{37}^{FM6}<\min\big\{X_{37}^{FM9},X_{37}^{OFM}\big\}\Big)=0.1749\ ,\ \overline{P}\Big(X_{37}^{FM6}<\min\big\{X_{37}^{FM9},X_{37}^{OFM}\big\}\Big)=0.3279$$

$$\underline{P}\Big(X_{37}^{OFM}<\min\big\{X_{37}^{FM6},X_{37}^{FM9}\big\}\Big)=0.2265\ ,\ \overline{P}\Big(X_{37}^{OFM}<\min\big\{X_{37}^{FM6},X_{37}^{FM9}\big\}\Big)=0.3808$$

Since

$$\underline{P}\left(X_{37}^{FM9}<\min\left\{X_{37}^{FM6},X_{37}^{OFM}\right\}\right)>\overline{P}\left(X_{37}^{FM6}<\min\left\{X_{37}^{FM9},X_{37}^{OFM}\right\}\right)$$

one could interpret the data as providing a strong indication that unit 37 is more likely to fail due to FM9 than due to FM6, in this setting with all other failure modes grouped into OFM. For example, if a person were to follow a subjective interpretation of lower and upper probabilities in terms of prices for desirable gambles, in line with Walley [13], then these lower and upper probabilities would imply that, for any price between 0.3279 and 0.3915, this person would be willing both to buy the gamble which pays 1 if unit 37 fails due to FM9 and to sell the gamble which pays 1 if unit 37 fails due to FM6. A quick look at the data may perhaps lead to some surprise that FM6 is not the more likely one to lead to failure, as it has caused relatively many early failures. However, one must not forget that it only caused failure of 7 out of the 36 units tested, the comparisons would be very different if the data were not competing risks data on the same units but completely independent failure times per group [20]. Similarly, a strong indication that unit 37 is more likely to fail due to FM9 than due to OFM can be claimed because

$$\underline{P}\left(X_{37}^{FM9}<\min\left\{X_{37}^{FM6},X_{37}^{OFM}\right\}\right)>\overline{P}\left(X_{37}^{OFM}<\min\left\{X_{37}^{FM6},X_{37}^{FM9}\right\}\right)$$

It is interesting to compare the results presented in Examples 2 and 3, as they illustrate some features that are very different in statistics using lower and upper probabilities when compared to methods using precise probabilities. The lower and upper probabilities for the event that unit 37 will fail due to FM9 are $[0.4358, 0.5804]$ in Example 2, where all other failure modes are grouped together, and $[0.3915, 0.5804]$ in Example 3, where FM6 is taken separately with all further failure modes grouped together. Hence, in the latter case, there is more imprecision in these upper and lower probabilities, while data are represented in more detail. This increase in imprecision, actually the fact that these upper and lower probabilities are nested with more imprecision if data are represented in more detail, is in line with a fundamental principle of NPI proposed and discussed by Coolen and Augustin [23] in the context of multinomial data. This leads to the conjecture that, for such competing risks data, if more failure modes are treated separately instead of grouped together, then lower and upper probabilities for an event that the next unit's failure is caused by a specific failure mode are nested, with imprecision increasing with the number of failure modes used. This conjecture has not been proven generally, due to the complexity of the expressions involved, but the authors strongly believe it to hold and all examples explored are in line with it. One could also have considered the question whether or not unit 37 will fail due to FM9 from a basic Bernoulli variables perspective, taking only into account that of 33 observed failures so far (neglecting the 3 units with right-censored lifetimes), 17 failed due to FM9. NPI for Bernoulli random quantities [24] leads to lower probability $17/34 = 0.5$ and upper probability $18/34 = 0.5294$ (note also that these bound the empirical probability $17/33 = 0.5152$), which lie inside the intervals created by the lower and upper probabilities for this event in Examples 2 and 3. This is also in line with the observation that a more detailed data representation leads to increased imprecision in the NPI approach. This Bernoulli data representation would, of course, not enable any inferences with regard to actual failure time.

The two NPI upper probabilities for the event that unit 37 will fail due to FM9, for the cases with all other failure modes grouped together (Example 2) and with FM6 separated (Example 3), are both equal to 0.5804. This is a consequence of the fact that this upper probability is realized with the extreme assignments of probability masses in the intervals created by the data in accordance to the lower survival function for FM9 and the upper survival function for the other failure modes. With all failure modes assumed to be independent, the upper survival function for the other failure modes combined is actually the same, whether or not FM6 is considered separately, this was discussed by Coolen, *et al* [17], who presented individual NPI lower and upper survival functions and also considered the data used in Examples 2 and 3, but they did not develop the NPI method for multiple comparisons that underlies the NPI method for competing risks presented in this paper.

To end discussion of Examples 2 and 3, it is useful to illustrate the NPI lower and upper survival functions that have been mentioned in these examples but which have not yet been presented. Figure 1 shows the NPI lower and upper survival functions for unit 37 for three situations, for which the upper survival functions are identical hence only the lower survival functions differ. The lower survival function $\underline{S}_{X_{37}}$ results from total neglection of the information on different failure modes, hence just by applying rc-$A_{(36)}$ [9] with 33 observed failure times and 3 right-censoring times. The lower survival function $\underline{S}^{2CR}_{X_{37}}$ corresponds to the situation with two (groups of) failure modes in Example 2, and is derived by multiplying the lower survival functions which are conditional on the given failure modes. Similarly, the lower survival function $\underline{S}^{3CR}_{X_{37}}$ corresponds to the situation with three (groups of) failure modes in Example 3. These lower and upper survival functions show a similar nested structure, related to the level of detail of the data representation, as was discussed above for the event that FM9 causes the failure of unit 37.

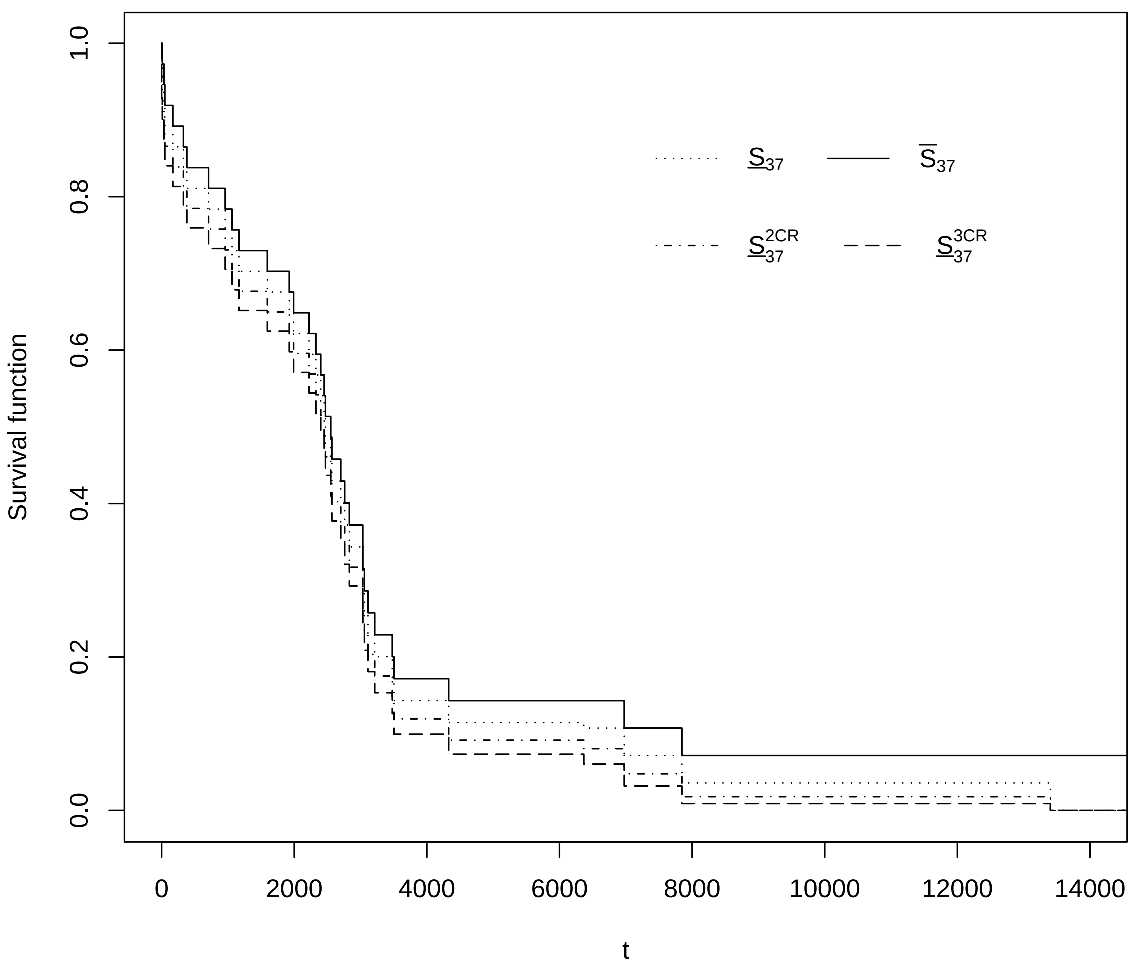

Figure 1: Lower and upper survival functions for unit 37

Figure 2 shows the NPI lower and upper survival functions for unit 37 conditioned on the specific failure mode, for FM9 and for FM6, corresponding to Example 3. For example, $\underline{S}^{FM9}_{X_{37}}$ and $\overline{S}^{FM9}_{X_{37}}$ are based on rc-$A_{(36)}$ applied with the data set with the 17 failure times related to failures caused by FM9 treated as actual failure time observations, and the other

19 observations in the data set as right-censored data, and similar for FM6. This figure nicely illustrates the effect of the relatively many early failures due to FM6, and the fact that there are far fewer failures due to FM6 than due to FM9 is reflected in far greater imprecision (the difference between corresponding upper and lower survival functions) at larger times. Note that, in the NPI approach based on rc-$A_{(n)}$, the lower survival function is always equal to zero beyond the largest observation, no matter if this is an observed failure time or a right-censored observation, while the upper survival function remains positive, this is discussed in more detail by Coolen and Yan [9].

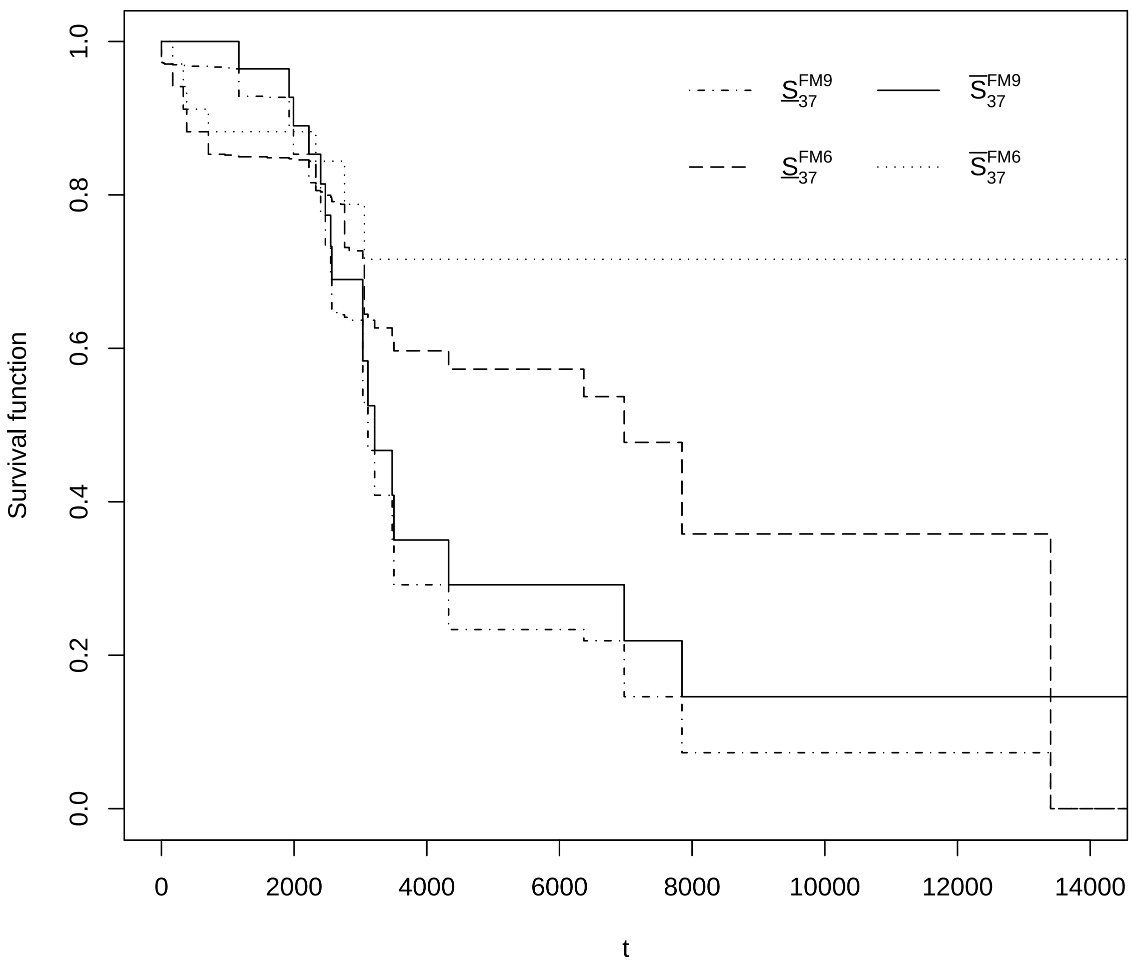

Figure 2: Lower and upper conditional survival functions for unit 37

5 Progressive Type-II censoring

By the nature of $A_{(n)}$, NPI is a frequentist statistical methodology [4, 6, 7], which however can be interpreted in a way similar to Bayesian statistics [5, 8]. An important advantage over more established frequentist methods is that NPI does not depend on counterfactuals, that is data which were not actually observed but could have been observed. For example, these are important in hypothesis testing, which has led to a large literature on frequentist methods for related problems considering slightly varying experimental procedures. In NPI, as in Bayesian statistics, the inferences only involve the actual data observed, although a warning is needed about the fact that, quite obviously, to apply NPI one must be happy with the exchangeability assumption on the data and future observation(s), which may be non-trivial depending on experimental set-up.

One topic that has led to a substantial literature in frequentist statistics involving right-censored data is progessive censoring [25, 26], where, during a lifetime experiment, non-failing units are withdrawn from the experiments. This could be done to save cost or time, but it may also be useful, at the moment a unit fails, to study the unit in detail in comparison with units in the same experiment that have not failed, to get better knowledge about the underlying cause of failure. Different progressive censoring schemes vary in the way that the number and times for such withdrawal of non-failing units are defined or decided upon, which are relevant with regard to the corresponding counterfactuals in hypothesis testing. For NPI, the specific nature of such progressive censoring schemes can reasonably be assumed to be irrelevant, as it just leads to further right-censored observations without any more specific information on the withdrawn units that would undermine the exchangeability assumption. Maturi, *et al* [27] introduced NPI for comparing two groups of lifetime data under progressive censoring schemes, with careful discussion of different schemes and comparison to other frequentist approaches for such data. They did not consider progressive censoring combined with competing risks data, which we briefly discuss in this section, and illustrate in an example which is based on Examples 2 and 3 in Section 4. The progressive censoring scheme considered here is known in the literature as 'progressive Type-II censoring' [25, 26], for other progressive censoring schemes one can follow the same approach, a flexibility which is one of the advantages of NPI when compared to the more established frequentist statistical methods.

In progressive Type-II censoring, at each failure time regardless of the failure cause, some randomly chosen non-failing units may be removed from the experiment. Adding such possible censored data to the competing risks scenario presented in this paper, the competing risks data per failure mode can consist of a number of observed failures caused by the specific failure mode considered, right-censored observations for failures caused by other failure modes, right-censored observations resulting from removing some non-failing units at failure times of other units (due to the progressive censoring scheme), and general right-censored observations due to unknown failure modes or other reasons, as was also allowed earlier in this paper. The key thing here is that right-censored data of any kind are dealt with in the same manner, per failure mode, in NPI for competing risks, so effectively there is no difference in the way NPI for competing risks data deals with right-censored data of the last two types discussed, which are right-censored observations for all failure modes. Also the manner in which these tied observations are dealt with is the same as discussed before.

Example 4

Suppose that, in the tests of the electrical appliances leading to the data in Example 2 and 3, it had been decided that, in order to learn more about the physics underlying common failure modes, 3 non-failing units were to be removed from the experiment as soon as the third failure due to the same failure mode occurs, enabling detailed comparison of the condition of the failed units with units that did not yet fail. Assume that the non-failing units withdrawn from the experiment are selected randomly from those still in the study. At time 381, when the third failure caused by FM6 occurs, three non-failing units would be withdrawn, hence leading to three right-censored observations at that time. Assume that the unit which in the original data (Table 2) failed at time 1990 due to FM9 was one of the three withdrawn at time 381, and that the unit failing at time 2223 is the third one failing due to FM9. Then a further three units are withdrawn at that moment to enable detailed study of the processes underlying FM9 through comparison with non-failed units. Suppose that this process leads to the data presented in Table 5, where as before right-censoring times are indicated by '-' for failure mode.

# cycles	FM	# cycles	FM	# cycles	FM
11	1	1167	9	2568	9
35	15	1594	2	2761	6
49	15	1925	9	2831	2
170	6	2223	9	3034	9
329	6	2223	-	3034	9
381	6	2223	-	3112	9
381	-	2223	-	3214	9
381	-	2327	6	3504	9
381	-	2400	9	4329	9
708	6	2471	9	6976	9
958	10	2551	9	7846	9
1062	5	2565	-	13403	-

Table 5: Failure data for electrical appliance test under progressive censoring

If, in analogy to Example 2, all failure modes other than FM9 are grouped together and jointly considered as one failure mode OFM, then the NPI lower and upper probabilities for the event that unit 37 will fail due to FM9 are

$$\underline{P}(X_{37}^{FM9} < X_{37}^{OFM}) = 0.4658, \quad \overline{P}(X_{37}^{FM9} < X_{37}^{OFM}) = 0.6258$$

Note that these NPI lower and upper probabilities are not nested when compared to those in Example 2, which is due to the fact that now the information per failure mode is really different. If, as in Example 3, failure modes FM9 and FM6 are considered separately, with all the other failure modes grouped as OFM, then the resulting NPI lower and upper probabilities for the events that unit 37 will fail due to FM9, due to FM6 or due to OFM, are

$$\underline{P}\big(X_{37}^{FM9}{<}\min\{X_{37}^{FM6}, X_{37}^{OFM}\}\big){=}0.4109\ ,\ \overline{P}\big(X_{37}^{FM9}{<}\min\{X_{37}^{FM6}, X_{37}^{OFM}\}\big){=}0.6258$$

$$\underline{P}\big(X_{37}^{FM6}{<}\min\{X_{37}^{FM9}, X_{37}^{OFM}\}\big){=}0.1668\ ,\ \overline{P}\big(X_{37}^{FM6}{<}\min\{X_{37}^{FM9}, X_{37}^{OFM}\}\big){=}0.3349$$

$$\underline{P}\big(X_{37}^{OFM}{<}\min\{X_{37}^{FM6}, X_{37}^{FM9}\}\big){=}0.1906\ ,\ \overline{P}\big(X_{37}^{OFM}{<}\min\{X_{37}^{FM6}, X_{37}^{FM9}\}\big){=}0.3593$$

These NPI lower and upper probabilities are again not nested in a specific general way with the NPI lower and upper probabilities in Example 3. However, they show the same nested behaviour as discussed in Section 4 with regard to the NPI lower and upper probabilities for the event $X_{37}^{FM9} < X_{37}^{OFM}$ in this setting with OFM including FM6.

6 Concluding remarks

In this paper, NPI for competing risks has been presented, with focus on the event that the next unit will fail due to a specific failure mode. Some specific properties and special cases are discussed and illustrated via examples in Section 4. NPI is a fully nonparametric statistical approach, which explicitly does not use any information or assumptions about the random quantities of interest other than the relevant $A_{(n)}$ and rc-$A_{(n)}$ assumptions per group, so here per failure mode. These inferences have a frequentist justification, but explicitly use the available data and do not require the use of counterfactuals, which for example happens in frequentist methods for hypothesis testing. As such, NPI is widely applicable and it is usually straightforward to implement different censoring scenarios, as briefly discussed and illustrated in Section 5 for a specific progressive censoring scheme. In many situations one may actually wish to use further relevant information, and opt for example for the use of Bayesian methods which allow prior information to be taken into account. Even in such cases, it may be useful to apply NPI together with other methods, as any differences in conclusions will result from the further assumptions underlying the other methods, hence NPI can give helpful insights into the influence of such further assumptions.

Appendix A

To prove the NPI lower and upper survival functions formulae (3) and (4), the following lemma is needed, for which some further notation is introduced. Let t_a, $a = 1, \ldots, n$ be n different ordered observations, each either a failure time ($\delta_a = 1$) or a right-censoring time ($\delta_a = 0$), and define $\delta_0 = 1$ corresponding to the definitions $t_0 = 0$ and $\tilde{n}_{t_0} = \tilde{n}_0 = n + 1$ introduced in Section 3.

Lemma 1
For all t_a, $a = 0, \ldots, n$,

$$(\tilde{n}_{t_a})^{\delta_a - 1} + \sum_{i=a+1}^{n} (\tilde{n}_{t_i})^{\delta_i - 1} \prod_{\{r: t_a \le c_r < t_i\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} = \tilde{n}_{t_a} \tag{11}$$

Proof of Lemma 1. The lemma is proven by induction as follows. First, for $t_a = t_n$ equation (11) is easily verfied both if t_n is failure time or a censoring time. Next, for $m = 0, 1, \ldots, n-2$, let $a = n - m$ and suppose that equation (11) holds for $t_a = t_{n-m}$,

$$(\tilde{n}_{t_{n-m}})^{\delta_{n-m} - 1} + \sum_{i=n-m+1}^{n} (\tilde{n}_{t_i})^{\delta_i - 1} \prod_{\{r: t_{n-m} \le c_r < t_i\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} = \tilde{n}_{t_{n-m}} \tag{12}$$

This implies that equation (11) also holds for $t_{a-1} = t_{n-m-1} = t_{n-(m+1)}$, which is shown now. The equality that needs to be proven is

$$(\tilde{n}_{t_{n-m-1}})^{\delta_{n-m-1} - 1} + \sum_{i=n-m}^{n} (\tilde{n}_{t_i})^{\delta_i - 1} \prod_{\{r: t_{n-m-1} \le c_r < t_i\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} = \tilde{n}_{t_{n-m-1}} \tag{13}$$

The left hand side of (13) can be written as

$$\begin{aligned}
&\left(\frac{1}{\tilde{n}_{t_{n-m-1}}}\right)^{1-\delta_{n-m-1}} + \left(\frac{\tilde{n}_{t_{n-m-1}} + 1}{\tilde{n}_{t_{n-m-1}}}\right)^{1-\delta_{n-m-1}} \left((\tilde{n}_{t_{n-m}})^{\delta_{n-m} - 1} + \sum_{i=n-m+1}^{n} (\tilde{n}_{t_i})^{\delta_i - 1} \prod_{\{r: t_{n-m} \le c_r < t_i\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} \right) \\
&= \left(\frac{1}{\tilde{n}_{t_{n-m-1}}}\right)^{1-\delta_{n-m-1}} + \left(\frac{\tilde{n}_{t_{n-m-1}} + 1}{\tilde{n}_{t_{n-m-1}}}\right)^{1-\delta_{n-m-1}} \tilde{n}_{t_{n-m}} \\
&= \left(\frac{1}{\tilde{n}_{t_{n-m-1}}}\right)^{1-\delta_{n-m-1}} + \left(\frac{\tilde{n}_{t_{n-m-1}} + 1}{\tilde{n}_{t_{n-m-1}}}\right)^{1-\delta_{n-m-1}} (\tilde{n}_{t_{n-m-1}} - 1) \\
&= \left(\frac{1}{\tilde{n}_{t_{n-m-1}}}\right)^{1-\delta_{n-m-1}} \left\{ 1 + \left(\tilde{n}_{t_{n-m-1}} + 1\right)^{1-\delta_{n-m-1}} (\tilde{n}_{t_{n-m-1}} - 1) \right\}
\end{aligned}$$

where the first equality follows from (12). Both if t_{n-m-1} is a failure time ($\delta_{n-m-1} = 1$) or a censoring time ($\delta_{n-m-1} = 0$), it follows straightforwardly that this expression is equal to $\tilde{n}_{t_{n-m-1}}$. Hence, by this induction argument equation (11) is proven to hold for all $a = 1, \ldots, n$. Finally, for $a = 0$, so $t_0 = 0$ for which $\delta_0 = 1$ and $\tilde{n}_{t_0} = \tilde{n}_0 = n + 1$ were defined, equation (11) follows directly by

$$
\begin{aligned}
&(\tilde{n}_0)^{\delta_0 - 1} + \sum_{i=1}^{n} (\tilde{n}_{t_i})^{\delta_i - 1} \prod_{\{r: t_0 \le c_r < t_i\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} \\
&= 1 + (\tilde{n}_{t_1})^{\delta_1 - 1} + \sum_{i=2}^{n} (\tilde{n}_{t_i})^{\delta_i - 1} \prod_{\{r: t_1 \le c_r < t_i\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} \\
&= 1 + \tilde{n}_{t_1} = 1 + n = \tilde{n}_{t_0}
\end{aligned}
$$

This completes the proof of Lemma 1.

Lemma 1, and indeed the NPI lower and upper survival functions (3) and (4), can also be interpreted along the same lines as the probability redistribution algorithm for right-censored data as introduced by Efron [28] and also discussed by Coolen and Yan [9].

Now Lemma 1 is used to prove formula (3) for the NPI lower survival function. To enable a single formula for all values of $t > 0$, let $t^i_{s_i+1} = t^{i+1}_0 = x_{i+1}$ for $i = 0, 1, \ldots, u - 1$. For $t \in [t^i_a, t^i_{a+1})$, the lower survival function, as given in [17], is equal to

$$
\begin{aligned}
\underline{S}_{X_{n+1}}(t) = \underline{S}_{X_{n+1}}(t^i_a) &= M_{X_{n+1}}(t^i_a, x_{i+1}) + \sum_{C(i,i^*,t^i_a)} M_{X_{n+1}}(t^i_{i^*}, x_{i+1}) \\
&= \frac{1}{n+1} \left\{ (\tilde{n}_{t^i_a})^{\delta^i_a - 1} \prod_{\{r: c_r < t^i_a\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} + \sum_{C(i,i^*,t^i_a)} (\tilde{n}_{t^i_{i^*}})^{\delta^i_{i^*} - 1} \prod_{\{r: c_r < t^i_{i^*}\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} \right\} \\
&= \frac{1}{n+1} \prod_{\{r: c_r < t^i_a\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} \left\{ (\tilde{n}_{t^i_a})^{\delta^i_a - 1} + \sum_{C(i,i^*,t^i_a)} (\tilde{n}_{t^i_{i^*}})^{\delta^i_{i^*} - 1} \prod_{\{r: t^i_a \le c_r < t^i_{i^*}\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} \right\} \\
&= \frac{1}{n+1} \prod_{\{r: c_r < t^i_a\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} \; \tilde{n}_{t^i_a}
\end{aligned}
$$

where $\sum_{C(i,i^*,t^i_a)}$ denotes the sums over all i from 0 to u and over all i^* from 0 to s_i such that $t^i_{i^*} > t^i_a$. Again, t^i_a can be failure time ($\delta^i_a = 1$) or a censoring time ($\delta^i_a = 0$).

Lemma 1 is also used to prove formula (4) for the NPI upper survival function [17], which, for $t \in [x_i, x_{i+1})$, is equal to

$$
\begin{aligned}
\overline{S}_{X_{n+1}}(t) &= M_{X_{n+1}}(x_i, x_{i+1}) + \sum_{C(i,i^*,x_i)} M_{X_{n+1}}(t^i_{i^*}, x_{i+1}) \\
&= \frac{1}{n+1} \prod_{\{r: c_r < x_i\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} \left\{ 1 + \sum_{C(i,i^*,x_i)} (\tilde{n}_{t^i_{i^*}})^{\delta^i_{i^*} - 1} \prod_{\{r: x_i \le c_r < t^i_{i^*}\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} \right\} \\
&= \frac{1}{n+1} \prod_{\{r: c_r < x_i\}} \frac{\tilde{n}_{c_r} + 1}{\tilde{n}_{c_r}} \, \tilde{n}_{x_i}
\end{aligned}
$$

where $\sum_{C(i,i^*,x_i)}$ denotes the sums over all i from 0 to u and over all i^* from 0 to s_i such that $t^i_{i^*} > x_i$.

Appendix B

The NPI lower and upper probabilities (7) and (8) are derived as the sharpest bounds, based on the relevant rc-$A_{(n)}$ assumptions, for the probability

$$P = P\left(X_{l,n+1} < \min_{\substack{1\le j\le k\\ j\ne l}} X_{j,n+1}\right) = \sum_{C(j,\,i_j)} P\left(X_{l,n+1} < \min_{\substack{1\le j\le k\\ j\ne l}} \{X_{j,n+1}\}, \bigcap_{\substack{j=1\\ j\ne l}}^{k} \{X_{j,n+1} \in (x_{j,i_j}, x_{j,i_j+1})\}\right)$$

where $\sum_{C(j,\,i_j)}$ denotes the sums over all i_j from 0 to u_j for $j = 1, \ldots, k$ but not including $j = l$. To derive these NPI lower and upper probabilities, the following lemma by Coolen and Yan [19] is needed, using M-functions as defined in Section 2.

Lemma 2
For $s \ge 2$, let $W_l = (w_l, r)$, with $w_1 < w_2 < \ldots < w_s < r$, so these are nested intervals $W_1 \supset W_2 \supset \ldots \supset W_s$ with the same right end-point r (which may be infinity). Consider two independent real-valued random quantities A and B. Let the probability distribution for A be partially specified via M-function values, with all probability mass $P(A \in W_1)$ described by the s M-function values $M_A(W_l)$, so $\sum_{l=1}^{s} M_A(W_l) = P(A \in W_1)$. Then, without additional assumptions, $\sum_{l=1}^{s} P(B < w_l)M_A(W_l) \le P(B < A, A \in W_1) \le P(B < r)P(A \in W_1)$, and these bounds are the maximum lower and minimum upper bounds that generally hold.

First consider the lower probability (7), which is derived as the sharpest general lower bound for the above probability P,

$$\begin{aligned} P &\ge \sum_{C(j,\,i_j,\,i_j^*)} P\left(X_{l,n+1} < \min_{\substack{1\le j\le k\\ j\ne l}} \{t^{i_j}_{j,i_j^*}\}\right) . \prod_{\substack{j=1\\ j\ne l}}^{k} M^j(t^{i_j}_{j,i_j^*}, x_{j,i_j+1}) \\ &\ge \sum_{C(j,\,i_j,\,i_j^*)} \left[\sum_{i_l=0}^{u_l} 1(x_{l,i_l+1} < \min_{\substack{1\le j\le k\\ j\ne l}} \{t^{i_j}_{j,i_j^*}\}) P^l(x_{l,i_l}, x_{l,i_l+1})\right] \prod_{\substack{j=1\\ j\ne l}}^{k} M^j(t^{i_j}_{j,i_j^*}, x_{j,i_j+1}) \end{aligned}$$

where $\sum_{C(j,\,i_j,\,i_j^*)}$ denotes the sums over all i_j^* from 0 to s_{j,i_j} and over all i_j from 0 to u_j for $j = 1, \ldots, k$ but not including $j = l$. The first inequality follows by putting all probability mass for each $X_{j,n+1}$ ($j = 1, \ldots, k$ and $j \ne l$) assigned to the intervals $(t^{i_j}_{j,i_j^*}, x_{j,i_j+1})$ ($i_j =$

$0, \ldots, u_j$ and $i_j^* = 0, 1, \ldots, s_{j,i_j}$) at the left end-points of these intervals, and by using the lemma presented above for the nested intervals. The second inequality follows by putting all probability mass for $X_{l,n+1}$ in each of the intervals $(t^{i_l}_{l,i_l^*}, x_{l,i_l+1})$ ($i_l = 0, \ldots, u_l$ and $i_l^* = 0, 1, \ldots, s_{l,i_l}$) at the right end-points of these intervals.

The derivation of the corresponding NPI upper probability (8) is given below. The first inequality follows by putting all probability mass for each $X_{j,n+1}$ ($j = 1, \ldots, k$ and $j \neq l$) assigned to the intervals $(t^{i_j}_{j,i_j^*}, x_{j,i_j+1})$ ($i_j = 0, \ldots, u_j$ and $i_j^* = 0, 1, \ldots, s_{j,i_j}$) at the right end-points of these intervals, and by using the lemma presented above for the nested intervals. The second inequality follows by putting all probability mass for $X_{l,n+1}$ in each of the intervals $(t^{i_l}_{l,i_l^*}, x_{l,i_l+1})$ ($i_l = 0, \ldots, u_l$ and $i_l^* = 0, 1, \ldots, s_{l,i_l}$) at the left end-points of these intervals.

$$\begin{aligned} \overline{P} &\leq \sum_{C(j,\, i_j)} P\left(X_{l,n+1} < \min_{\substack{1 \leq j \leq k \\ j \neq l}} \{x_{j,i_j+1}\}\right) \prod_{\substack{j=1 \\ j \neq l}}^{k} P^j(x_{j,i_j}, x_{j,i_j+1}) \\ &\leq \sum_{C(j,\, i_j)} \left[\sum_{i_l=0}^{u_l} \sum_{i_l^*=0}^{s_{l,i_l}} 1(t^{i_l}_{l,i_l^*} < \min_{\substack{1 \leq j \leq k \\ j \neq l}} \{x_{j,i_j+1}\}) M^l(t^{i_l}_{l,i_l^*}, x_{l,i_l+1})\right] \prod_{\substack{j=1 \\ j \neq l}}^{k} P^j(x_{j,i_j}, x_{j,i_j+1}) \end{aligned}$$

where, as before, $\sum_{C(j,\, i_j)}$ denotes the sums over all i_j from 0 to u_j for $j = 1, \ldots, k$ but not including $j = l$.

Appendix C

In this appendix the derivation of the NPI lower and upper probabilities (9) and (10) is presented, and the corresponding imprecision is derived. In the case of two competing risks, the NPI lower probability (7) becomes

$$\underline{P}^{(l)} = \underline{P}(X_{l,n_l+1} < X_{j,n_j+1}) = \sum_{i_j=0}^{u_j} \sum_{i_j^*=0}^{s_{j,i_j}} \left\{\sum_{i_l=0}^{u_l} 1(x_{l,i_l+1} < t^{i_j}_{j,i_j^*}) P^l(x_{l,i_l}, x_{l,i_l+1})\right\} M^j(t^{i_j}_{j,i_j^*}, x_{j,i_j+1})$$

Let R_l (R_j) be the set of ranks of all failure times due to failure mode l (j). That is $R_l \subset \{1, 2, \ldots, n\}$ and $R_j = \{1, 2, \ldots, n\} \setminus R_l$, where it is assumed that all n units considered have actually failed due to one of these two failure modes. As any failure of a unit due to failure mode l leads to a right-censored observation for failure mode j for that unit, and vice versa, $x_{l,(r_l)} = c_{j,(r_l)}$ ($x_{j,(r_j)} = c_{l,(r_j)}$) for $r_l \in R_l$ ($r_j \in R_j$). Let $\sum_{C\left(i_j, i_j^*, c_{j,(r_l)}\right)}$ denote the sums over all i_j from 0 to u_j and over all i_j^* from 0 to s_{j,i_j} such that $t^{i_j}_{j,i_j^*} \geq c_{j,(r_l)}$. Then the NPI lower probability (9) can be written as

$$
\begin{aligned}
\underline{P}^{(l)} &= \sum_{i_j=0}^{u_j} \sum_{i_j^*=0}^{s_{j,i_j}} \left\{ \sum_{i_l=0}^{u_l} 1(x_{l,i_l+1} < t^{i_j}_{j,i_j^*}) P^l(x_{l,i_l}, x_{l,i_l+1}) \right\} M^j(t^{i_j}_{j,i_j^*}, x_{j,i_j+1}) \\
&= \sum_{r_l \in R_l} P^l(x_{l,(r_l-1)}, x_{l,(r_l)}) \sum_{C\left(i_j, i_j^*, c_{j,(r_l)}\right)} M^j(t^{i_j}_{j,i_j^*}, x_{j,i_j+1}) \\
&= \sum_{r_l \in R_l} P^l(x_{l,(r_l-1)}, x_{l,(r_l)}) \, \underline{S}_{X_{j,n+1}}(c_{j,(r_l)}) \\
&= \sum_{r_l \in R_l} \left(\frac{1}{n+1} \prod_{\{r: c_{l,r} < x_{l,(r_l)}\}} \frac{\tilde{n}_{c_{l,r}} + 1}{\tilde{n}_{c_{l,r}}} \right) \left(\frac{1}{n+1} \tilde{n}_{c_{j,(r_l)}} \prod_{\{r: c_{j,r} < c_{j,(r_l)}\}} \frac{\tilde{n}_{c_{j,r}} + 1}{\tilde{n}_{c_{j,r}}} \right) \\
&= \sum_{r_l \in R_l} \left(\frac{1}{n+1} \right)^2 \left(\frac{n+1}{n+2-r_l} \right) (n+1-r_l) \\
&= \frac{1}{n+1} \sum_{r_l \in R_l} \frac{n+1-r_l}{n+2-r_l}
\end{aligned}
$$

The fifth equality in this derivation results from the fact that, with all units assumed to fail due to one of the two failure modes considered, and $x_{l,(r_l)} = c_{j,(r_l)}$ and $x_{j,(r_j)} = c_{l,(r_j)}$ for all $r_l \in R_l$ and $r_j \in R_j$, the two product terms combine into a single product over all first $r_l - 1$ observations. This product simplifies to $\frac{n+1}{n+2-r_l}$, and $\tilde{n}_{c_{j,(r_l)}} = n + 1 - r_l$ completes the justification of the fifth equality.

The corresponding NPI upper probability (10) can be derived similarly, but it is easier to do so by use of the conjugacy property, by which

$$
\overline{P}^{(l)} = 1 - \underline{P}(X_{l,n_l+1} > X_{j,n_j+1}) = 1 - \underline{P}^{(j)}
$$

where

$$
\underline{P}^{(j)} = \frac{1}{n+1} \sum_{r_j \in R_j} \frac{n+1-r_j}{n+2-r_j}
$$

is of course obtained directly from the above expression for $\underline{P}^{(l)}$).

Finally, the fact that for this situation with two failure modes and all n units failing due to one of them, the imprecision is constant, follows by

$$\begin{aligned}\text{Imprecision} &= 1 - \left\{\underline{P}^{(l)} + \underline{P}^{(j)}\right\} \\ &= 1 - \frac{1}{n+1}\left\{\sum_{r_l \in R_l} \frac{n+1-r_l}{n+2-r_l} + \sum_{r_j \in R_j} \frac{n+1-r_j}{n+2-r_j}\right\} \\ &= 1 - \frac{1}{n+1}\sum_{i=1}^{n} \frac{n+1-i}{n+2-i} \\ &= \frac{1}{n+1}\left[1 + \sum_{i=1}^{n} \frac{1}{n+2-i}\right] \\ &= \frac{1}{n+1}\sum_{i=1}^{n+1} \frac{1}{i}\end{aligned}$$

Appendix D: Notation

A, B	random quantities in Lemma in Appendix B.
$A_{(n)}$	Hill's inferential assumption.
$c_1, \ldots, c_v$	right-censored observations.
$c_{j,1}, \ldots, c_{j,v_j}$	right-censored observations considering FMj.
$c_1^i, \ldots, c_{s_i}^i$	right-censored observations in (x_i, x_{i+1}), $i = 0, \ldots, u$.
$c_{j,1}^{i_j}, \ldots, c_{j,s_{j,i_j}}^{i_j}$	right-censored observations in (x_{j,i_j}, x_{j,i_j+1}), $i_j = 0, \ldots, u_j$.
$\delta_{i^*}^{i}$ $(\delta_{i_j^*}^{i_j})$ (δ_a)	1 if $t_{i^*}^{i}$ $(t_{i_j^*}^{i_j})$ (t_a) is a failure time (or is 0), 0 if a right-censoring time.
FMj	failure mode j.
k	number of distinct failure modes.
m	an index used in Appendix A
$M_Y(a, b)$	M function: probability mass for Y assigned to interval (a, b).
M^j	M function for X_{j,n_j+1}.
n	number of units on which data is available.
$\tilde{n}_t$	number of units in the risk set just prior to time t, and $\tilde{n}_0 = n + 1$.
P_j	NPI probability for X_{j,n_j+1}.
$\underline{P}, \overline{P}$	lower and upper probability, respectively.
r	right-end point of nested intervals in Lemma in Appendix B.
rc-$A_{(n)}$	Coolen and Yan's assumption right-censoring $A_{(n)}$.
$\underline{S}, \overline{S}$	lower and upper survival function, respectively.
s	number of nested intervals in Lemma in Appendix B.
s_i	number of right-censored observations in (x_i, x_{i+1}), $i = 0, \ldots, u$.
s_{j,i_j}	number of right-censored observations in (x_{j,i_j}, x_{j,i_j+1}), $i_j = 0, \ldots, u_j$.

Appendix D: Notation (Cont.)

T	minimum of $T_1, \dots, T_k$.
T_j	unit's random time to failure under condition that failure occurs due to FMj.
t_a	time of observed event, either failure time (or time 0) or right-censoring ($a = 1, \dots, n$), and $t_0 = 0$.
$t^i_{i^*}$	time of observed event in $[x_i, x_{i+1})$, either failure time (or time 0) or right-censoring.
$t^{i_j}_{j,i^*_j}$	time of observed event in $[x_{j,i_j}, x_{j,i_j+1})$, either failure time caused by FMj (or time 0) or right-censoring.
u (u_j)	number of observed failure times (considering FMj).
v (v_j)	number of right-censored observations (considering FMj).
W_l	nested intervals in Lemma in Appendix B.
w_l	left-end points of nested intervals in Lemma in Appendix B.
X_{n+1} ($X_{j,n+1}$)	failure time of one future unit (under condition that failure occurs due to FMj).
$x_1, \dots, x_u$	observed failure times.
x_0, $x_{j,0}$	equal to 0.
x_{u+1}, x_{j,u_j+1}	equal to ∞.
x_{j,i_j}	observed failure time, failure caused by FMj.
R_l (R_j)	the set of ranks of all failure times due to failure mode l (j)
Y	random quantity used in definition of M-function.
$Y_1, \dots, Y_{n+1}$	random quantities used in general formulation of $A_{(n)}$.
$y_1, \dots, y_n$	ordered observations of $Y_1, \dots, Y_n$.
y_0	equal to 0.

References

[1] Bedford, T., Alkali, B. and Burnham, R. (2008). Competing risks in reliability. In: E.L. Melnick, B.S. Everitt (Eds.), *Encyclopedia of Quantitative Risk Analysis and Assessment*, pp. 307-312. Chichester: Wiley.

[2] Ray, M.R. (2008). Competing risks. In: E.L. Melnick, B.S. Everitt (Eds.), *Encyclopedia of Quantitative Risk Analysis and Assessment*, pp. 301-307. Chichester: Wiley.

[3] Tsiatis, A. (1975). A nonidentifiability aspect of the problem of competing risks. *Proceedings of the National Academy of Sciences of the United States of America* **72**, 20-22.

[4] Augustin, T. and Coolen, F.P.A. (2004). Nonparametric predictive inference and interval probability. *Journal of Statistical Planning and Inference* **124**, 251-272.

[5] Coolen, F.P.A. (2006). On nonparametric predictive inference and objective Bayesianism. *Journal of Logic, Language and Information* **15**, 21-47.

[6] Hill, B.M. (1968). Posterior distribution of percentiles: Bayes' theorem for sampling from a population. *Journal of the American Statistical Association* **63**, 677-691.

[7] Hill, B.M. (1988). De Finetti's theorem, induction, and $A_{(n)}$ or Bayesian nonparametric predictive inference (with discussion). In: J.M. Bernardo, et al. (Eds.), *Bayesian Statistics 3*, pp. 211-241. Oxford: Oxford University Press.

[8] Hill, B.M. (1993). Parametric models for A_n: splitting processes and mixtures. *Journal of the Royal Statistical Society B* **55**, 423-433.

[9] Coolen, F.P.A. and Yan, K.J. (2004). Nonparametric predictive inference with right-censored data. *Journal of Statistical Planning and Inference* **126**, 25-54.

[10] Coolen, F.P.A. and Utkin, L.V. (2008). Imprecise reliability. In: B. Everitt and E. Melnick (Eds.), *Encyclopedia of Quantitative Risk Analysis and Assessment*, pp. 875-881. Chichester: Wiley.

[11] Utkin, L.V. and Coolen, F.P.A. (2007). Imprecise reliability: an introductory overview. In: G. Levitin (Ed.), *Computational Intelligence in Reliability Engineering, Volume 2: New Metaheuristics, Neural and Fuzzy Techniques in Reliability*, pp. 261-306. New York: Springer.

[12] De Finetti, B. (1974). *Theory of Probability*. Chichester: Wiley.

[13] Walley, P. (1991). *Statistical Reasoning with Imprecise Probabilities*. London: Chapman and Hall.

[14] Weichselberger, K. (2000). The theory of interval-probability as a unifying concept for uncertainty. *International Journal of Approximate Reasoning* **24**, 149-170.

[15] Weichselberger, K. (2001). *Elementare Grundbegriffe einer allgemeineren Wahrscheinlichkeitsrechnung I. Intervallwahrscheinlichkeit als umfassendes Konzept*. Heidelberg: Physika.

[16] Yan, K.J. (2002). *Nonparametric Predictive Inference with Right-Censored Data*. PhD thesis, University of Durham (available online from www.npi-statistics.com).

[17] Coolen, F.P.A., Coolen-Schrijner, P. and Yan, K.J. (2002). Nonparametric predictive inference in reliability. *Reliability Engineering and System Safety* **78**, 185-193.

[18] Shafer, G. (1976). *A Mathematical Theory of Evidence*. Princeton: Princeton University Press.

[19] Coolen, F.P.A. and Yan, K.J. (2003). Nonparametric predictive comparison of two groups of lifetime data. In: J.M. Bernard, T. Seidenfeld, M. Zaffalon (Eds.), *ISIPTA'03: Proceedings of the Third International Symposium on Imprecise Probability: Theory and Applications*, pp 148-161. Waterloo (Canada): Carleton Scientific.

[20] Maturi, T.A., Coolen-Schrijner, P. and Coolen, F.P.A. (2009). Nonparametric predictive comparison of lifetime data with early termination of experiments. In submission.

[21] Maturi, T.A., Coolen-Schrijner, P. and Coolen, F.P.A. (2009). Nonparametric predictive selection with early experiment termination. In submission.

[22] Lawless, J.F. (2003). *Statistical Models and Methods for Lifetime Data* (2nd ed.). New York: Wiley.

[23] Coolen, F.P.A. and Augustin, T. (2009). A nonparametric predictive alternative to the Imprecise Dirichlet Model: the case of a known number of categories. *International Journal of Approximate Reasoning* **50**, 217-230.

[24] Coolen, F.P.A. (1998). Low structure imprecise predictive inference for Bayes' problem. *Statistics & Probability Letters* **36**, 349-357.

[25] Balakrishnan, N. and Aggarwala, R. (2000) *Progressive Censoring : Theory, Methods, and Applications.* Boston: Birkhäuser.

[26] Kundu, D., Kannan, N. and Balakrishnan, N. (2004). Analysis of progressively censored competing risks data. In: N. Balakrishnan, C.R. Rao (Eds.), *Handbook of Statistics 23: Advances in Survival Analysis*, pp 331-348. Amsterdam: Elsevier.

[27] Maturi, T.A., Coolen-Schrijner, P. and Coolen, F.P.A. (2009). Nonparametric predictive comparison of lifetime data under progressive censoring. *Journal of Statistical Planning and Inference*, to appear (DOI: 10.1016/j.jspi.2009.07.027).

[28] Efron, B. (1967). The two sample problem with censored data. In: *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, Vol 4, pp 831-853. New York: Prentice-Hall.